Research Statement

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My research is in combinatorial algebraic geometry, which is a field at the interaction of combinatorics, algebra, and geometry. In particular, I study toric varieties, a class of geometric spaces which are studied using combinatorial objects like polytopes and polyhedral fans (§1). The main goal of my research is to understand the relationship between the combinatorial structure of toric varieties and the algebraic and combinatorial structure of their cohomology (§2). Many problems related to my research are great for undergraduate research because they can attract a range of students with various mathematical interests, they provide numerous, accessible entry points from algebra, combinatorics, or geometry, and they can be explored by using or creating mathematical software (§4).

1. Overview of research interests

The overarching theme of my research is to study the algebraic and combinatorial structure of the cohomology of toric varieties. Toric varieties are geometric objects with inherent combinatorial structure: every polyhedral fan $\Sigma \subseteq \mathbb{R}^n$ defines a toric variety X_Σ, and every toric variety arises in this way. Because toric varieties can be defined via combinatorial data, toric varieties are a rich source of geometric models for combinatorial objects. Moreover, the combinatorial structure of toric varieties makes them an important testing ground in algebraic geometry, as many algebraic invariants can be explicitly computed in terms of the fan Σ and its embedding in $\mathbb{R}^n$. Cohomology theories are important examples of such algebraic invariants.

I study two types of cohomology theories on toric varieties: the singular cohomology of the toric variety itself and the sheaf cohomology of toric vector bundles over the toric variety.

For singular cohomology, my focus is on the case of smooth and non-compact toric varieties. Whereas compact and potentially-singular toric varieties have well-understood cohomology rings, a correct description of the multiplicative structure in the smooth, non-compact case only comes from recent work of Franz, who described these rings in terms of Tor algebras [Fra21]. My main interest is how the combinatorial structure of a smooth toric variety interacts with the combinatorial structure and algebraic Lefschetz properties of its singular cohomology ring. Recently, I have had success in using tools from combinatorial commutative algebra to study the singular cohomology of smooth toric varieties and to develop a good notion of Lefschetz properties in this context [Bin26a, Bin26b, BV26, Bin].

On the other hand, my main interest in the sheaf cohomology of toric vector bundles is to understand the combinatorial structure of higher sheaf cohomology. For line bundles on toric varieties, there is an established theory of using rational polytopes to study cohomological properties of a line bundle [CLS11, Chapter 6]. There is a similar story now for toric vector bundles on toric varieties [DRJS18]. Instead of using a single polytope, cohomological properties of toric vector bundles are studied using a finite collection of polytopes, called a *parliament of polytopes*. When the parliament is chosen carefully, integer points of the polytopes in the parliament compute the global sections, or H^0, of the vector bundle; however, it is still unknown how to combinatorially understand the higher cohomology groups. In work with my collaborators, we give a method for using the parliament to compute the higher cohomology of toric vector bundles over projective space $\mathbb{P}^n$ [BCD+].

Matroids are a recurring character in not only my research but also in the field of combinatorial algebraic geometry. A matroid is a combinatorial object that abstracts the notions of dependence common to linear algebra and graph theory, and matroids have applications across mathematics and computer science, such as in hyperplane arrangements, coding theory, and combinatorial optimization.

Along with studying general toric varieties, I also study particular classes of toric varieties that come from matroids. Here, a matroid M produces the *Bergman fan* Σ_M [AK06], and therefore a toric variety X_{Σ_M}. The geometric theory of matroids studies combinatorial properties of matroids through the toric varieties X_{Σ_M} and powerful tools in algebraic geometry. The most famous example of this comes from Adiprasito, Huh, and Katz [AHK18] who used the Chow ring of X_{Σ_M} along with tools from positivity in algebraic geometry to prove two long-standing conjectures in matroid theory. My work on the singular cohomology rings of matroids seeks to generalize the Chow ring of a matroid to the larger singular cohomology ring, and in the case of uniform matroids, I showed interesting parallels between these cohomology theories of matroids [Bin26b, BV26].

Matroids also show up in tropical toric vector bundles, which are a recent combinatorial abstraction of toric vector bundles [KM24, KM26]. Tropical toric vector bundles generalize the combinatorial structure found in the parliaments of polytopes approach to studying toric vector bundles and are defined in terms of matroids. Part of my interest in studying the combinatorics of higher cohomology for tropical vector bundles is to define a notion of higher cohomology for tropical toric vector bundles.

2. Past and current research

2.1. Singular cohomology of smooth toric varieties. My work on the singular cohomology of smooth toric varieties focuses on the relation between the structure of the singular cohomology of X_Σ and the structure of the defining fan $\Sigma \subseteq \mathbb{R}^n$. More specifically, I am interested in the question of which aspects of the singular cohomology ring of X_Σ depend only on the combinatorics of Σ and not its specific embedding in $\mathbb{R}^n$.

A full answer to this question is subtle when X_Σ is not compact. The major difficulty is that the incidence structure of cones in Σ alone does not determine the dimensions of the cohomology groups—the actual angles at which the cones intersect matter for the Betti numbers. Therefore, the entire singular cohomology group of a smooth toric variety is not *combinatorially determined*. However, I showed that a piece of the singular cohomology still is combinatorially determined [Bin]. The key insight is to use the associated gradeds of the weight filtration $W_\bullet H^\bullet(X_\Sigma)$ and the simplicial cohomology of Σ and its full subfans.

Theorem 1. *Let X_Σ be a smooth toric variety. Then the top non-vanishing weight of cohomology is*

$$\mathrm{Gr}^W_{2n-2i} H^\bullet(X_\Sigma) \cong \bigoplus_{\substack{I \subseteq \Sigma_1 \\ |I| = |\Sigma_1| - i}} \tilde{H}^\bullet(\Sigma|_I) \tag{1}$$

where i is the smallest integer for which (1) is non-trivial.

I also refined this result representation-theoretically by studying the action of the toric automorphism group $\mathrm{Aut}_T(X_\Sigma)$ on the top non-vanishing weight cohomology. It turns out that the isomorphism (1) is equivariant after twisting the right-hand side by an explicit 1-dimensional representation. Using this description, I showed that $\mathrm{Gr}^W_{2n-2i} H^\bullet(X_\Sigma)$ is only combinatorially determined as a vector space, not as an $\mathrm{Aut}_T(X_\Sigma)$-representation.

2.2. **Singular cohomology rings of matroids.** Another aspect of my work on the singular cohomology of smooth toric varieties is studying the singular cohomology of a combinatorially rich class of smooth toric varieties. In particular, I study the *singular cohomology ring* of a matroid, which is the singular cohomology of the toric variety X_{Σ_M} associated to the Bergman fan of a matroid M. This generalizes the well-studied *Chow ring* of a matroid [AHK18], as the Chow ring of a smooth toric variety is always a subring of the singular cohomology.

For general matroids, I showed some of the combinatorial structure that these singular cohomology rings possess.

Theorem 2 ([Bin26a, Theorem 6.6] and [Bin])**.** *Let M be a loopless matroid of rank r on ground set $[n]$. Then*

(1) $\mathrm{Gr}^W_j H^k(X_{\Sigma_M}) = 0$ for $j > n - r + k$ and $j < 2k - 2r + 2$.
(2) $H^k(X_{\Sigma_M}) = 0$ for $k > n + r - 2$.

As an $\mathrm{Aut}(M)$-representation, $\mathrm{Gr}^W_{2n-2} H^{n+r-2}(X_{\Sigma_M}) \cong \mathrm{OS}^r(M) \otimes \mathrm{sgn}$, where $\mathrm{OS}^\bullet$ is the Orlik–Solomon algebra of M.

The last part of this result is surprising because the Orlik–Solomon algebra is yet another cohomology theory for matroids that is not *a priori* related to the singular cohomology ring.

For the special class of uniform matroids, I gave a complete combinatorial description of the singular cohomology ring, and I showed surprising parallels between the Chow ring and the singular cohomology ring related to algebraic Lefschetz properties and real-rooted generating functions.

Lefschetz properties in commutative algebra study when multiplication by a special element—a hyperplane class—in a graded ring is always an injection or surjection between homogeneous components. One important property of the Chow ring a matroid is that it satisfies the Hard Lefschetz property. This says that multiplication by powers of a generic hyperplane class produces an isomorphism between cohomology in complementary degrees. For the singular cohomology ring of a matroid, I exhibited a slightly weaker *strong Lefschetz* property.

Theorem 3 ([Bin26b, Theorem 5.5])**.** *Let $\ell \in \mathrm{Gr}^W_2 H^2(X_{\Sigma_{U_{r,n}}})$ be a generic hyperplane class. Then the map*

$$\times\ell^d\colon\ \mathrm{Gr}^W_j H^k(X_{\Sigma_{U_{r,n}}}) \longrightarrow \mathrm{Gr}^W_{j+2d} H^{k+2d}(X_{\Sigma_{U_{r,n}}})$$

is injective for $0 \le d \le r + j - 2k - 1$ and surjective for $d > r + j - 2k + 1$.

This is the result I am most proud of, as it is the only sort of Lefschetz property known for a non-compact variety, and it requires a shift in perspective from using the standard grading of the cohomology ring to using the associated gradeds of the weight filtration.

Another interesting feature of the Chow ring of a uniform matroid is that its generating function has only real-roots, and it is an important conjecture that this is true for any matroid [FS24, Ste21]. For the singular cohomology ring of a matroid, the correct construction for a generating function is the family of *refined Hodge–Poincaré polynomials*

$$\underline{\mathrm{H}}^i_{U_{r,n}}(x) = \sum_{j=0}^{r-1} \dim \mathrm{Gr}^W_{2i+2j} H^{2j+i}(X_{\Sigma_{U_{r,n}}}) \cdot x^j.$$

These too turn out to be real-rooted.

Theorem 4 ([BV26])**.** *The refined Hodge–Poincaré polynomials of a uniform matroid are real-rooted.*

2.3. **Tropical toric vector bundles and sheaf cohomology.** Another cohomology theory on toric varieties that I work with is the sheaf cohomology of toric vector bundles. Toric varieties are a special class of geometric objects that are acted on by algebraic torus, $(\mathbb{C}^*)^n$. A toric vector bundle on X_Σ is a vector bundle $\mathscr{E}$ along with an action of the torus such that the structure map $\mathscr{E} \to X_\Sigma$ respects the torus actions. Such bundles were characterized by Kaneyama [Kan75] in terms of a cocycle condition and Klyachko [Kly89] in terms of a compatible family of decreasing filtrations of a vector space.

A recent development in this theory is a combinatorial abstraction of toric vector bundles called *tropical toric vector bundles*, defined by Khan–Maclagan [KM26] and Kaveh–Manon [KM24]. Tropical bundles may be thought of as replacing the decreasing filtrations of a vector space in Klyachko's description with a decreasing chain of flats of a matroid. Currently, tropical toric vector bundles have well-defined notions of certain cohomological invariants, like equivariant K- and Chern-classes, Euler characteristic, and rank of global sections, but not much else.

My research on tropical toric vector bundles has focused on providing a well-defined, combinatorial theory of sheaf cohomology. One major obstacle in defining this theory is that there is not a good notion of maps between tropical toric vector bundles, so the classical definition of sheaf cohomology does not apply. However, in joint work, I showed that there is at least well-defined notion of Betti numbers for the sheaf cohomology of tropical toric vector bundles over complex projective space $\mathbb{P}^n$.

Theorem 5 ([BCD+])**.** *Tropical toric vector bundles on $\mathbb{P}^n$ have a well-defined notion of Betti numbers, assuming the underlying matroid is modular. Moreover,*

(1) The alternating sum of Betti numbers computes the Euler characteristic.
(2) A tropical toric vector bundle $\mathfrak{E}$ over $\mathbb{P}^n$ splits if and only if, for every toric line bundle $\mathfrak{L}$, the higher Betti numbers of $\mathfrak{E} \otimes \mathfrak{L}$ vanish.

The last statement is a tropical analogue of Horrocks' splitting criterion, a well-known condition to determine when a vector bundle over $\mathbb{P}^n$ decomposes into a sum of simpler line bundles.

3. Future projects

The results I outlined in Section 2 form the basis of my research program to understand the algebraic and combinatorial structure of cohomology on toric varieties. In this section I describe some of the directions and plans I have for future work.

3.1. **Lefschetz properties on non-compact toric varieties.** Lefschetz properties for the singular cohomology of compact toric varieties have attracted a lot of attention recently in connection to the proof of the g-theorem [Adi19]. However, there is still much to discover about Lefschetz properties in the case of non-compact toric varieties, and my work in Theorem 3 gives a starting point.

Research Project 6. *Which classes of smooth, non-compact toric varieties exhibit Lefschetz properties?*

As not every smooth toric variety has a cohomology ring with Lefschetz properties, an answer to Problem 6 must then involve determining structural properties of fans and toric varieties which force Lefschetz properties. One promising structural property is that the toric variety should have a large collection of nice torus-invariant divisors that generate cohomology. In fact, this is the heart of Theorem 3, and this sort of argument is very similar to the "partition of unity" argument of Adiprasito [Adi19]. Getting a combinatorial description of such toric varieties would be very interesting.

Another approachable problem along the same lines of Problem 6 is the following.

Research Project 7. *Suppose that Σ' is a subdivision of the fan $\Sigma \subseteq \mathbb{R}^n$. Do X_Σ and $X_{\Sigma'}$ possess the same Lefschetz properties?*

A similar question for toric varieties satisfying a set of much stronger cohomological properties was answered in the affirmative by [ADH23]. We suspect that our work on toric blow-ups in [Bin26a] will lead to a clear answer.

3.2. **Singular cohomology of braid matroids and projective geometries.** Our work on uniform matroids shows that the singular cohomology rings of matroids have rich combinatorial structure that extends the well-studied combinatorics in the Chow ring. One goal of our research program is to understand these combinatorics for all matroids.

Long-Term Goal 8. *For an arbitrary matroid, give a combinatorial presentation or description of the singular cohomology ring.*

Achieving this goal would also provide a better understanding of the Chow ring of a matroid.

As a series of stepping-stones to achieving this long-term goal, we plan on considering the singular cohomology of other explicit families of matroids with nice symmetry.

Research Project 9. *Describe the singular cohomology rings of braid matroids and projective geometries.*

Our techniques for the singular cohomology of uniform matroids relied on the fact that the class of uniform matroids is closed under the combinatorial operations of truncation and restriction. As the classes of braid matroids and projective geometries also behave well under these operations, we expect to solve Problem 9 using generalizations of our techniques for uniform matroids.

3.3. **Tropical toric vector bundles on permutohedral varieties.** We would like to extend our work on the cohomology of tropical toric vector bundles over projective space to tropical toric vector bundles over a general base.

Long-Term Goal 10. *Develop a combinatorial notion of higher cohomology for tropical toric vector bundles.*

Such a cohomology theory seems far off, as the current candidates for maps between tropical toric vector bundles do not have the necessary additive structure to define cohomology like in the classical case. A more attainable interim goal is to define higher cohomology over other families of toric varieties with well-understood structure.

Research Project 11. *Define higher cohomology for tropical toric vector bundles over the permutohedral toric variety X_E.*

A promising approach to this problem, and its main motivation, is to combinatorialize the methods used in [Eur24, Liu25] to compute the cohomology of tautological bundles of matroids, which are important objects in geometric matroid theory [BEST23]. The idea there is to use the deletion map $f\colon X_E \to X_{E\setminus n}$, pushforwards, and the Leray spectral sequence to reduce the computation to the simpler computation of cohomology for bundles over $\mathbb{P}^1$. The key to combinatorializing this process is to compute the pushforward of a toric vector bundle in terms of the Klyachko data.

Research Project 12. *Let $f\colon X_\Sigma \to X_{\Sigma'}$ be a toric morphism, and let $\mathscr{E}$ be a toric vector bundle on X_Σ such that $f_*\mathscr{E}$ is a vector bundle. Describe the Klyachko data for $f_*\mathscr{E}$ in terms of that for $\mathscr{E}$.*

4. Research with Undergraduates

My research sets me up well to effectively guide undergraduate research. As my research touches various areas in combinatorics, algebra, and geometry, I can recruit and work with students with a wide range of interests and mathematical backgrounds. Here I give some examples of undergraduate research projects that relate to my own research.

4.1. **h-vectors of matroids.**

The collection of *independent sets* of a matroid forms a simplicial complex with special structure. Stanley showed that the h-vectors of these simplicial complexes are *O-sequences*, and famously conjectured that they are *pure O-sequences* [Sta77]. This conjecture is known in special cases but is still open [DLKK12, Mer01, Sch10]. Later, Chari conjectured a stronger statement that these h-vectors are *M-shellable* pure O-sequences [Cha97]. There are many classes of matroids for which Stanley's conjecture is known but Chari's conjecture is still open.

Undergraduate Project 13. *Strengthen proofs of special cases of Stanley's conjecture to special cases of Chari's conjecture.*

This problem is very combinatorial with deep connections to algebra, and students only need to know about posets at the beginning of this project. I am currently working on this project in a semester-long research course with 2 undergraduates and 2 early-PhD students. I expect us to get publishable results, with many of follow-up projects still remaining for future students.

4.2. **Structural properties of polyhedral fans and cohomology of smooth toric varieties.**

Part of my research plan involves connecting properties of the singular cohomology of X_Σ with combinatorial properties of Σ. There are a number of problems in this direction which are suitable for undergraduates interested in combinatorial topology. One known connection between fans and singular cohomology characterizes the *Cohen–Macaulayness* of the fan Σ with the general shape of the singular cohomology of X_Σ. Being Cohen–Macaulay is a topological condition on fans, and Baclawski [Bac82] introduced stronger notions of being k-Cohen–Macaulay which should also admit a characterization via the singular cohomology of X_Σ.

Undergraduate Project 14. *Give necessary and sufficient conditions for a fan Σ to be k-Cohen–Macaulay in terms of the singular cohomology of X_Σ.*

While the background needed to solve this problem is only some linear algebra and familiarity with simplicial complexes, a student would likely need some abstract algebra and topology to appreciate and be interested in the result.

4.3. **Mathematical software.**

Many problems in combinatorial algebraic geometry use mathematical software like `Macaulay2` or `Sage` to compute and explore examples, and there is always a need to develop programs that implement new ideas in the field.

Undergraduate Project 15. *Explore Lefschetz properties of smooth toric varieties by writing mathematical software.*

As this kind of project requires a close reading of a few math paper, they are a great way to prepare students for future research in the area or for graduate school.

4.4. **Reading courses.** Reading courses are a great way to engage a wide range of students who may not currently have the interest or background needed for other forms of undergraduate research. I have extensive experience in reading math books with students through Directed Reading Programs (DRPs) in graduate school, and I have mentored many students in diverse topics. One of my favorite topics was with a student who was interested in the connection between math and art, and we explored chip-firing on infinite graphs and the patterns that they produce.

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