Research Statement

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1. Introduction

Toric varieties provide a fascinating source of interactions between algebraic geometry and combinatorics. For one, toric varieties are geometric objects with intrinsic combinatorial structure. Every normal toric variety is determined by a rational polyhedral fan, and many geometric invariants of the toric variety can be computed in terms of the combinatorics of the fan. This turns geometric problems into combinatorial ones. Conversely, toric varieties turn combinatorics into interesting geometry. Combinatorial objects like polytopes and atomic lattices induce rational polyhedral fans and therefore toric varieties. These toric varieties often remember pieces of the original combinatorial structure, and this gives a way to import powerful tools from algebraic geometry into combinatorics.

The rich connections between geometry and combinatorics on toric varieties have recently led to a number of exciting developments, particularly involving various cohomology theories. Starting with [AHK18], the use of Chow cohomology on toric varieties associated to matroids has resulted in the resolution of numerous long-standing conjectures in matroid theory; see also [ADH23, BST23]. Adiprasito's proof of the g-conjecture for PL spheres essentially studies Lefschetz properties for the intersection cohomology of simplicial toric prevarieties [Adi19]. These are both variants of the singular cohomology of toric varieties. Other cohomology theories on toric varieties—sheaf cohomology and K-theory—have also recently grown into important tools in the geometric theory of matroids [BEST23, EHL23, EFL25, Eur24]. Even in algebraic geometry, there has been a renewed effort to recast descriptions of toric vector bundles into more combinatorial language and to produce combinatorial criteria for geometric properties of toric vector bundles [DRJS18, KM26, Dev23].

My research focuses on these relations between combinatorics and cohomology on toric varieties. In particular, I study singular cohomology and sheaf cohomology on toric varieties, where I am interested in combinatorial descriptions of cohomology and the cohomology of combinatorially-defined toric varieties and vector bundles.

(1) **Singular cohomology of toric varieties.** Previous work on the singular cohomology of toric varieties has typically focused on the complete (and potentially singular) case. My research on the other hand focuses on the smooth (and potentially non-complete) case. The combinatorics here is more subtle than in the complete case but still very rich. For instance, see §2.1.1 for my work on a combinatorial description of the top non-vanishing weight piece of cohomology. Most of my work on the singular cohomology rings of toric varieties focuses on toric varieties which are combinatorially-defined via matroids (§2.1.2). My results show strong analogies between these singular cohomology rings and the well-studied Chow rings of matroids, particularly in terms of real-rooted generating functions and Lefschetz properties. These results suggest that the cohomology of smooth toric varieties is a promising source of interesting combinatorics and Lefschetz properties, and I propose a few projects along these lines in §§3.1 and 3.2.

(2) **Tropical toric vector bundles.** Tropical toric vector bundles are a recent combinatorial and tropical abstraction of equivariant vector bundles on a toric variety. The main motivation for studying tropical toric vector bundles is to define a good notion of vector bundles

on a class of tropical varieties and to provide a theory for tautological bundles of non-realizable matroids [BEST23]. My main interest in this topic centers around developing a combinatorial notion of sheaf cohomology for these tropical vector bundles by extracting the combinatorial data used in the classic algebro-geometric setting. I describe some results on the sheaf cohomology of tropical toric vector bundles over $\mathbb{P}^n$ in §2.2, and I propose a related project for bundles over the permutohedral toric variety in §3.3.

2. Previous Work

2.1. **Singular cohomology of smooth toric varieties.** We give a brief overview of the singular cohomology rings of smooth toric varieties in order to describe my previous research. Let X_Σ be a smooth toric variety defined by the rational fan $\Sigma \subseteq N_{\mathbb{Q}} \cong \mathbb{Q}^n$ with dense open torus T. The singular cohomology ring of X_Σ is strongly related to combinatorial commutative algebra. Indeed, Franz–Fu exhibit a natural isomorphism of rings

$$H^\bullet(X_\Sigma) \cong \operatorname{Tor}_\bullet^{\operatorname{Sym} N_{\mathbb{Q}}^\vee}(\mathbb{Q}[\Sigma], \mathbb{Q}), \tag{1}$$

where $\mathbb{Q}[\Sigma]$ is the *Stanley–Reisner* ring of Σ [FF24]. Moreover, the natural bigrading of Tor has a geometric meaning. Under the *weight filtration* $0 = W_0 H^\bullet(X_\Sigma) \leq \cdots \leq W_{2n} H^\bullet(X_\Sigma) = H^\bullet(X_\Sigma)$ of Deligne's mixed Hodge structure, Weber showed that $\operatorname{Tor}_i^{\operatorname{Sym} N_{\mathbb{Q}}^\vee}(\mathbb{Q}[\Sigma], \mathbb{Q})_j \cong \operatorname{Gr}_{2j}^W H^{2j-i}(X_\Sigma)$ [Web04]. In other words, the bigraded *Betti numbers* of $\mathbb{Q}[\Sigma]$ are the *Hodge numbers* of X_Σ.

2.1.1. *Top non-vanishing weight cohomology of smooth toric varieties.* The relationship between the singular cohomology ring of X_Σ and the combinatorics of Σ is subtle. For example, when X_Σ is complete, the Hodge numbers are combinatorially determined by Σ and independent of the embedding in $N_{\mathbb{Q}}$. Here, they correspond to the *h-vector* of the associated simplicial complex $\Delta(\Sigma)$.[1] When X_Σ is not complete this does not hold, and the Hodge numbers depend on the particular embedding $\Sigma \subseteq N_{\mathbb{Q}}$.

However, even though the entire singular cohomology ring of X_Σ is not combinatorially determined by Σ, some pieces of cohomology still are. We show in [Bin] that the top non-vanishing weight of cohomology depends only on the simplicial cohomology of $\Delta(\Sigma)$ and its subcomplexes.

Theorem 1. *Let X_Σ be a smooth toric variety. Then the top non-vanishing weight of cohomology is*

$$\operatorname{Gr}_{2n-2i}^W H^\bullet(X_\Sigma) \cong \bigoplus_{\substack{I \subseteq \Delta(\Sigma)_0 \\ |I| = |\Delta(\Sigma)_0| - i}} \tilde{H}^\bullet(\Delta(\Sigma)|_I) \tag{2}$$

where i is the smallest integer for which (2) is non-trivial and $\Delta(\Sigma)_0$ is the set of vertices of $\Delta(\Sigma)$.

Previously, it was only known that the *top-weight* cohomology $\operatorname{Gr}_{2n}^W H^\bullet(X_\Sigma)$ is combinatorially determined, using work of Hacking [Hac08] and Payne [Pay13] on boundary complexes and Alexander duality. When the top-weight cohomology vanishes, boundary complexes are not enough to see any of the other associated gradeds. Our approach instead uses (1) and flat base change of Tor.

In [Bin] we also study the action of the toric automorphism group $\operatorname{Aut}_T(X_\Sigma)$ (which permutes rays in Σ, so may be viewed as a subgroup of $\mathfrak{S}_{\Delta(\Sigma)_0}$) on the top non-vanishing weight cohomology and show that the isomorphism (2) is equivariant after twisting the right-hand side by $\det \operatorname{Cl}(X_\Sigma)_{\mathbb{Q}}$ and

[1] This $\Delta(\Sigma)$ can be viewed as the simplicial complex of the intersection $\Sigma \cap S^{n-1}$ or the link of the origin in Σ.

a sign representation. Using this description, we show that $\mathrm{Gr}^W_{2n-2i} H^\bullet(X_\Sigma)$ is only combinatorially determined as a vector space, not as an $\mathrm{Aut}_T(X_\Sigma)$-representation.

Corollary 2. *We construct two rational fans $\Sigma_1, \Sigma_2 \subseteq N_\mathbb{Q}$ such that*

(1) $\Delta(\Sigma_1) = \Delta(\Sigma_2)$
(2) As subgroups of $\mathfrak{S}_{\Delta(\Sigma_1)_0}$, $\mathrm{Aut}_T(X_{\Sigma_1}) = \mathrm{Aut}_T(X_{\Sigma_2})$.
(3) $\mathrm{Gr}^W_{2n} H^\bullet(X_{\Sigma_1})$ *and* $\mathrm{Gr}^W_{2n} H^\bullet(X_{\Sigma_2})$ *are not isomorphic as* $\mathrm{Aut}_T(X_{\Sigma_1})$*-representations.*

2.1.2. *Singular cohomology rings of matroids.* Much of my work on the singular cohomology of toric varieties studies particular toric varieties arising from matroids, which are combinatorial objects that abstract the notion of linear independence in vector spaces. In their field-defining paper, Adiprasito, Huh, and Katz [AHK18] introduce the *Chow ring* of a matroid M to be the Chow ring of the toric variety X_{Σ_M} associated to the order complex of the lattice of flats of M. These toric varieties and their Chow rings have since played a central role in the geometric theory of matroids.

However, the Chow ring misses a lot of the geometry of X_{Σ_M}. Indeed, X_{Σ_M} is generally not compact, so the Chow ring injects, but does not surject, into the singular cohomology ring. In [Bin26a], I began studying the *singular cohomology ring* of a matroid, $H^\bullet(X_{\Sigma_M})$, in order to understand these toric varieties better. My main result shows that these singular cohomology rings contain interesting combinatorial structure.

Theorem 3 ([Bin26a, Theorem 6.6] and [Bin])**.** *Let M be a loopless matroid of rank r on ground set $[n]$. Then*

(1) $\mathrm{Gr}^W_j H^k(X_{\Sigma_M}) = 0$ *for $j > n - r + k$ and $j < 2k - 2r + 2$.*
(2) $H^k(X_{\Sigma_M}) = 0$ *for $k > n + r - 2$.*

As an $\mathrm{Aut}(M)$*-representation,* $\mathrm{Gr}^W_{2n-2} H^{n+r-2}(X_{\Sigma_M}) \cong \mathrm{OS}^r(M) \otimes \mathrm{sgn}$*, where* $\mathrm{OS}^\bullet$ *is the* Orlik–Solomon algebra *of M.*

For uniform matroids, I was able to give a complete description of the singular cohomology ring, and I showed that this ring has a number of interesting combinatorial properties ranging from Lefschetz properties to real-rooted generating functions [Bin26b].

One of the miraculous properties of the Chow ring of a matroid is that it satisfies the *Kähler package*, and in particular, Hard Lefschetz. This property says that multiplication by powers of a generic hyperplane class induces isomorphisms between cohomology in complementary degree. However, the Kähler package for the singular cohomology ring fails due to the dimension of the top-degree cohomology often not being 1 (this is a consequence of Theorem 3). Still, we show that the singular cohomology rings of uniform matroids satisfy a slightly weaker *strong Lefschetz property* with respect to the weight filtration.

Theorem 4 ([Bin26b, Theorem 5.5])**.** *Let $\ell \in \mathrm{Gr}^W_2 H^2(X_{\Sigma_{U_{r,n}}})$ be a generic hyperplane class. Then the map*

$$\times \ell^d \colon \mathrm{Gr}^W_j H^k(X_{\Sigma_{U_{r,n}}}) \longrightarrow \mathrm{Gr}^W_{j+2d} H^{k+2d}(X_{\Sigma_{U_{r,n}}})$$

is injective for $0 \leq d \leq r + j - 2k - 1$ and surjective for $d > r + j - 2k + 1$.

To our knowledge, this is the only sort of Lefschetz result known for non-compact toric varieties.

Another surprising feature of the Chow ring of a uniform matroid is that its generating function—the *Chow polynomial*—is real-rooted, and it is an important conjecture that the Chow polynomial of any matroid is real-rooted [FS24, Ste21]. For the singular cohomology ring, the correct analogues of the Chow polynomial are the *refined Hodge–Poincaré polynomials*

$$\underline{\mathrm{H}}^i_{U_{r,n}}(x) = \sum_{j=0}^{r-1} \dim \mathrm{Gr}^W_{2i+2j} H^{2j+i}(X_{\Sigma_{U_{r,n}}}) \cdot x^j$$

for $0 \leq i \leq n-r$. These polynomials are the "higher weight" analogues of the Chow polynomial, with $\underline{\mathrm{H}}^0_{U_{r,n}}(x)$ being the Chow polynomial itself.

Theorem 5 ([BV])**.** *The refined Hodge–Poincaré polynomials of a uniform matroid are real-rooted.*

The proof uses the combinatorics of an explicit basis of the singular cohomology ring I constructed in [Bin26b] along with a criterion for polynomials to be real-rooted from [BV25, Theorem 4.16].

2.2. **Tropical toric vector bundles and sheaf cohomology.** A toric vector bundle on X_Σ is a vector bundle $\mathscr{E}$ such that the structure map $\mathscr{E} \to X_\Sigma$ is T-equivariant. Such bundles were characterized by Kaneyama [Kan75] in terms of a cocycle condition and Klyachko [Kly89] in terms of a compatible family of decreasing filtrations of a vector space.

An exciting recent development in this theory is a combinatorial and tropical abstraction of toric vector bundles called *tropical toric vector bundles*, defined independently and simultaneously by Khan–Maclagan [KM26] and Kaveh–Manon [KM24]. Tropical bundles may be thought of as replacing the decreasing filtrations of a vector space in Klyachko's description with a decreasing chain of flats of a matroid. Currently, tropical toric vector bundles have well-defined notions of equivariant K- and Chern-classes, Euler characteristic, and rank of global sections.

My research on tropical toric vector bundles has focused on providing a well-defined, combinatorial theory of sheaf cohomology. One major obstacle in defining this theory is that there is not yet an agreed-upon notion of morphisms for tropical bundles, so there is no way of using a Čech complex for cohomology as in the classical case of toric vector bundles. However, in joint work with Sergio Cristancho, Caitlin Davis, Christopher Manon, Emma Pickard, and Mahrud Sayrafi, we construct a well-defined notion of Betti numbers for the sheaf cohomology of tropical toric vector bundles over $\mathbb{P}^n$. The main idea is that the various formulas for the Betti numbers of toric vector bundles on $\mathbb{P}^n$ given by Klyachko [Kly89] can all be defined matroid-theoretically, and then one is left to check that these formulas are well-behaved.

Theorem 6 ([BCD+])**.** *Klyachko's formulas are well-defined for tropical toric vector bundles on $\mathbb{P}^n$, assuming the underlying matroid is modular. Moreover,*

(1) The alternating sum of Betti numbers computes the Euler characteristic.
(2) (Horrocks' splitting criterion) A tropical toric vector bundle $\mathfrak{E}$ over $\mathbb{P}^n$ splits if and only if, for every toric line bundle $\mathfrak{L}$, the higher Betti numbers of $\mathfrak{E} \otimes \mathfrak{L}$ vanish.

3. Future Projects

3.1. **Lefschetz properties on non-compact toric varieties.** Lefschetz properties on the Artinian and Gorenstein reductions of Stanley–Reisner rings has attracted much attention recently in connection to the g-theorem [Adi19]. Morally speaking, these reductions correspond to the cohomology of *toric prevarieties* whose minimal orbits are fixed points, which is a generalization of compact toric varieties. However, my work on Theorem 4 suggests there is also much to discover about Lefschetz properties in the case of non-compact toric varieties.

Problem 7. *Which classes of smooth (non-complete) toric varieties exhibit Lefschetz properties?*

Answers to this question would enhance our knowledge of the geometry of non-compact toric varieties as well give interesting examples of Lefschetz properties in combinatorial commutative algebra.

While every projective toric variety exhibits Hard Lefschetz, there are examples of quasi-projective toric varieties which exhibit no Lefschetz properties. For example, a fan whose maximal cones have wildly different dimensions often will not exhibit Lefschetz properties. An answer to Problem 7 must then involve determining structural properties of fans and toric varieties which force Lefschetz properties.

One potential structural property that appears promising in this regard is the toric variety having a large collection of nice torus-invariant divisors. In fact, the argument at the heart of Theorem 4 relies on the fact that each $X_{\Sigma_{U_{r,n}}}$ possesses a set of torus-invariant divisors D_i which (1) look like projective space times a torus, and (2) generate the reduced singular cohomology under the Gysin morphisms $H^\bullet(D_i) \to \tilde{H}^\bullet(X_{\Sigma_{U_{r,n}}})$. This sort of argument is very similar to the "partition of unity" argument of Adiprasito [Adi19], and getting a combinatorial description of such toric varieties would be very interesting.

Another approachable problem along the same lines of Problem 7 is the following.

Problem 8. *Suppose that Σ' is a subdivision of the rational fan $\Sigma \subseteq N_{\mathbb{Q}}$. Do X_Σ and $X_{\Sigma'}$ possess the same Lefschetz properties?*

A similar question for toric varieties satisfying the Kähler package, along with a few other properties, was answered in the affirmative by [ADH23], but the question is open for weaker Lefschetz properties. We suspect that our work on toric blow-ups in [Bin26a] will lead to a clear answer.

3.2. **Singular cohomology of braid matroids and projective geometries.** Our work on uniform matroids shows that the singular cohomology rings of matroids have rich combinatorial structure that extends the well-studied combinatorics in the Chow ring. One goal of our research program is to understand these combinatorics for all matroids.

Long-Term Goal 9. *For an arbitrary loopless matroid, give combinatorial formulas for the refined Hodge–Poincaré polynomials of the singular cohomology ring and produce a combinatorial basis.*

Aside from understanding the singular cohomology ring better, achieving this goal would give a better understanding of the Chow ring and Chow polynomial of a matroid. Current formulas for the Chow polynomials of a general matroid are recursive and do not give much combinatorial insight into the matroid itself. In our work on the singular cohomology rings of uniform matroids, the extra information contained in the singular cohomology ring allowed us to construct a natural basis for the Chow ring that gave combinatorial proofs for many of the existing formulas for the Chow polynomials of uniform matroids. We are hopeful that the additional information captured by the singular cohomology ring yields similar combinatorial structure for the Chow polynomial of a general matroid.

As a series of stepping-stones to achieving this long-term goal, we plan on considering the singular cohomology of other explicit families of matroids with nice symmetry.

Problem 10. *Describe the singular cohomology rings of braid matroids and projective geometries.*

Our techniques for the singular cohomology of uniform matroids relied on the fact that the class of uniform matroids is closed under truncation and restriction. As the classes of braid matroids and projective geometries also behave well under these operations, we expect to solve Problem 10 using generalizations of our techniques for uniform matroids.

3.3. **Tropical toric vector bundles on permutohedral varieties.** We would like to extend our work on the cohomology of tropical toric vector bundles over projective space to tropical toric vector bundles over a general base.

Long-Term Goal 11. *Develop a combinatorial notion of higher cohomology for tropical toric vector bundles.*

At the moment, developing such a cohomology theory seems far off, as the obvious morphisms for tropical toric vector bundles do not have abelian structure. For example, Jun, Sistko, and Wright [JSW25] construct a nice category for tropical toric reflexive sheaves using morphisms from the theory of matroids over idylls [BL21]. However, this category is only *proto*-abelian, and derived functor cohomology is subtle here [FLS17]. In future work, it may be profitable to apply ideas from cohomology in $\mathbb{F}_1$-geometry to resolve these issues.

A more attainable interim goal is to define higher cohomology over families of toric varieties with well-understood structure.

Problem 12. *Define higher cohomology for tropical toric vector bundles over the* permutohedral toric variety X_E.

A promising approach to this problem, and in fact its main motivation, is to combinatorialize the methods used in [Eur24, Liu25] to compute the cohomology of tautological bundles of matroids. The idea there is to use the deletion map $f\colon X_E \to X_{E\setminus n}$, derived pushforwards, and the Leray spectral sequence to reduce the computation to the cohomology of bundles over $\mathbb{P}^1$. The key to combinatorializing this process is to compute the pushforward of a toric vector bundle in terms of the Klyachko data.

Problem 13. *Let $f\colon X_\Sigma \to X_{\Sigma'}$ be a toric morphism, and let $\mathscr{E}$ be a toric vector bundle on X_Σ such that $f_*\mathscr{E}$ is a reflexive sheaf. Describe the Klyachko data for $f_*\mathscr{E}$ in terms of that for $\mathscr{E}$.*

We suspect that one can apply the ideas from [Zot25], which describes an algorithm for computing higher direct images in terms of Cox modules, in order to solve this problem. Aside from helping to resolve Problem 12, an interesting and important application of Problem 13 would be to understand the combinatorics of the Frobenius pushforward of vector bundles on toric varieties. This functor on toric vector bundles plays a key role in the theory of toric vector bundles on $\mathbb{P}^n$ [Rąc25].

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