REAL-ROOTEDNESS FOR THE SINGULAR COHOMOLOGY OF UNIFORM MATROIDS

1. Introduction

The Chow ring $A^\bullet(M)$ of a matroid M is one of the central objects in modern matroid theory. Introduced through the work of Feichtner and Yuzvinsky, it has played a fundamental role in the development of combinatorial Hodge theory and in the proof of several long-standing conjectures concerning the characteristic polynomial and the number of independent sets of a matroid [FY04]. A key feature of the Chow ring is that it satisfies the Kähler package: Poincaré duality, the Hard Lefschetz theorem, and the Hodge–Riemann relations [AHK18]. The Hilbert–Poincaré series of the Chow ring of a matroid M is known as its *Chow polynomial*. The Kähler package implies that the Chow polynomial is palindromic and unimodal. The following conjecture has been proposed independently by Huh–Stevens [Ste21] and Ferroni–Schröter [FS24].

Conjecture 1.1. The Chow polynomial of every matroid has only real zeros.

Since a polynomial with nonnegative coefficients and only real zeros has a log-concave, and hence unimodal, coefficient sequence, real-rootedness would place the known numerical properties of the Chow ring in a stronger and more rigid framework. The real-rootedness conjecture remains open for general matroids, but it has been established for several important families. In particular, the Chow polynomials of uniform matroids was proven to be real-rooted by Brändén and the second author [BV26]. Related real-rootedness results are also known for augmented Chow polynomials [FMSV24] and for Chow polynomials associated with several classes of geometric lattices and graded posets [BV25, CFL25, HS25], in the sense of [FMV26].

In [Bin26a], the first author recently introduced the *singular cohomology ring* of a loopless matroid M. It is defined as the singular cohomology ring $H^\bullet(X_{\Sigma_M})$ of the smooth toric variety associated with the Bergman fan Σ_M or, equivalently, as the homology of the Koszul complex of the Stanley–Reisner ring $\mathbb{Q}[\Sigma_M]$ viewed as an algebra over $\operatorname{Sym} N_{\mathbb{Q}}^\vee$, where $N_{\mathbb{Q}}^\vee$ is the rationalization of the character lattice of X_{Σ_M}. This construction extends the ordinary Chow ring, as $H_0(K(\mathbb{Q}[\Sigma_M]))$ is naturally isomorphic to $A^\bullet(M)$.

The singular cohomology ring and Koszul homology groups carry a bigrading corresponding to the associated gradeds of the weight filtration

$$\operatorname{Gr}^W_{2j} H^{2j-i}(X_{\Sigma_M}) \cong H_i\big(K(\mathbb{Q}[\Sigma_M])\big)_j,$$

and their dimensions are encoded by the *refined Hodge–Poincaré polynomials*

$$\mathsf{H}^i_M(x) = \sum_j \dim_{\mathbb{Q}} H_i\big(K(\mathbb{Q}[\Sigma_M])\big)_j \cdot x^{j-i}.$$

The Chow polynomial is recovered when $i = 0$. Unlike the Chow ring, however, the full singular cohomology ring does not satisfy the usual Kähler package. In particular, its refined Hodge–Poincaré polynomials need not be palindromic.

For a uniform matroid $\mathsf{U}_{r,n}$ of rank r on ground set $[n] = \{1, \ldots, n\}$, the first author constructed an explicit basis for the singular cohomology ring in terms of *retral chains* and *weakly retral chains* of flats together with certain admissible exterior-algebra elements [Bin26b]. This basis leads to explicit formulas for the refined Hodge–Poincaré polynomials. He also proved that the singular cohomology ring of a uniform matroid satisfies a version of the strong Lefschetz property. As a consequence, each refined Hodge–Poincaré polynomial $\mathsf{H}^i_{\mathsf{U}_{r,n}}(x)$ is unimodal, despite the failure of the ordinary Kähler package, and he conjectured that each $\mathsf{H}^i_{\mathsf{U}_{r,n}}(x)$ has only real zeros.

The purpose of this paper is to resolve this conjecture.

Theorem 1.2. *For every $1 \leq r \leq n$ and every $0 \leq i \leq n - r$, the refined Hodge–Poincaré polynomial $\mathsf{H}^i_{\mathsf{U}_{r,n}}(x)$ has only real zeros.*

The theorem also extends the real-rootedness result for the Chow polynomial of a uniform matroid, which is precisely the case $i = 0$. This provides evidence for the following.

Conjecture 1.3. For every loopless matroid M of rank r on ground set $[n]$ and for every $0 \leq i \leq n - r$, the refined Hodge–Poincaré polynomial $\mathsf{H}^i_M(x)$ has only real zeros.

2. Combinatorics of weakly retral chains

In this section we provide a brief overview of the *retral* and *weakly retral* chains of flats in $\mathscr{L}(\mathsf{U}_{r,n})$ used in [Bin26b] to construct a basis for the singular cohomology of a uniform matroid in terms of Koszul homology. We then describe a new construction for pairs of weakly retral chains and *admissible wedges*; this leads to a new formula for the refined Hodge–Poincaré polynomials which we will use in §4 to deduce real-rootedness.

2.1. Definition of (weakly) retral chains and admissible wedges.

Given a chain of proper flats $\mathscr{F}$ in $\mathscr{L}(\mathsf{U}_{r,n})$, we define the *ranks* of $\mathscr{F}$ to be

$$\operatorname{rks}(\mathscr{F}) = \{\operatorname{rk}(F) : F \in \mathscr{F}\}.$$

We will follow the convention in [Bin26b] that all chains of flats contain $\emptyset$, or equivalently, $0 \in \operatorname{rks}(\mathscr{F})$. For flats $F, G \in \mathscr{L}(M)$, we say that G *covers* F if $F < G$ and there is no flat H such that $F < H < G$. If M is a uniform matroid and G is a proper flat, G covers F if and only if $|G \setminus F| = 1$.

Definition 2.1. Let $\mathscr{F} = \{\emptyset = F_0 < \cdots < F_s \neq [n]\}$ be a chain of proper flats in $\mathscr{L}(\mathsf{U}_{r,n})$. We say that $\mathscr{F}$ is *retral* if

(Cover condition): For $1 \leq i \leq s$, whenever F_i covers F_{i-1}, $F_i \setminus F_{i-1} = \max(F_{i+1} \setminus F_{i-1})$.

Here we use the convention that $F_{s+1} = [n]$.

In other words, retrality is a lexicographic maximality condition on the covering relations in a chain of flats. For chains that include a hyperplane (a corank 1 flat), we define *weakly retral* chains by removing this cover condition on the covering relations involving the hyperplane.

Definition 2.2. Let $\mathscr{F} = \{\emptyset = F_0 < \cdots < F_s \neq [n]\}$ be a chain of proper flats in $\mathscr{L}(\mathsf{U}_{r,n})$. We say that $\mathscr{F}$ is *weakly retral* if

(1) F_s is a hyperplane, and
(2) For $1 \leq i < s$, whenever F_i covers F_{i-1}, $F_i \setminus F_{i-1} = \max(F_{i+1} \setminus F_{i-1})$.

For the rational vector space N with basis $\{\ell_i : i \in [n]\}$, let $N_{\mathbb{Q}}^{\vee}$ be the subspace perpendicular to $\ell_1 + \cdots + \ell_n$ under the canonical inner product. Thus, $N_{\mathbb{Q}}^{\vee}$ is spanned by the elements $\ell_u - \ell_v$ such that $u, v \in [n]$.

To every weakly retral chain $\mathscr{F}$ in $\mathscr{L}(\mathsf{U}_{r,n})$, we associate a set of *admissible wedges* in the exterior algebra $\bigwedge^{>0} N_{\mathbb{Q}}^{\vee}$.

Definition 2.3. Let $\mathscr{F} = \{\emptyset = F_0 < \cdots < F_{s-1} < F_s \neq [n]\}$ be a weakly retral chain in $\mathscr{L}(\mathsf{U}_{r,n})$. An *admissible wedge* to $\mathscr{F}$ is a basic wedge

$$(\ell_{u_1} - \ell_{v_1}) \wedge \cdots \wedge (\ell_{u_i} - \ell_{v_i}) \in \bigwedge^{i>0} N_{\mathbb{Q}}^{\vee}$$

such that

(1) $\{u_1 < \cdots < u_i\} \subseteq [n] \setminus F_s$,
(2) each v_k is the successor of u_k in the ordered set $[n] \setminus F_s$, and
(3) if F_s covers F_{s-1}, then $u_1 < (F_s \setminus F_{s-1})$.

These retral and weakly retral chains define the following basis for the singular cohomology ring of a uniform matroid.

Theorem 2.4 ([Bin26b, Theorem 3.11]). *The set*

$$\left\{ \prod_{F \in \mathscr{F} \setminus \emptyset} x_F : \mathscr{F} \text{ is retral} \right\} \cup \left\{ \prod_{F \in \mathscr{F} \setminus \emptyset} x_F \otimes \xi : \mathscr{F} \text{ is weakly retral, } \xi \text{ admissible to } \mathscr{F} \right\}$$

is a basis of the singular cohomology ring $H_\bullet(K(\mathbb{Q}[\Sigma_{\mathsf{U}_{r,n}}])) \cong H^\bullet(X_{\Sigma_{\mathsf{U}_{r,n}}})$.

2.2. **Counts of weakly retral chains and admissible wedges.** We now give a method for counting and constructing the following class of weakly retral chains and admissible wedges.

Definition 2.5. For $0 \leq j \leq r-1$ and $1 \leq i \leq n-r$, let

$$W_{\{0,j,j+1,\ldots,r-1\},i}(\mathsf{U}_{r,n}) = \left\{ (\mathscr{F}, \xi) : \begin{array}{c} \mathscr{F} \text{ is weakly retral, } \xi \in \wedge^i N_{\mathbb{Q}}^{\vee} \text{ admissible to } \mathscr{F}, \\ \text{and } \operatorname{rks}(\mathscr{F}) = \{0, j, j+1, \ldots, r-1\} \end{array} \right\}.$$

For ease of notation, we will write this as $W_{j,i}(\mathsf{U}_{r,n})$. Note that $W_{0,i}(\mathsf{U}_{r,n}) = W_{1,i}(\mathsf{U}_{r,n})$.

Along with the derangement polynomials $d_j(x)$ (see §3.1 for a definition), the cardinalities of $W_{j,i}(\mathsf{U}_{r,n})$ provide formulas for the refined Hodge–Poincaré polynomials.

Lemma 2.6. *For $i \geq 1$,*

$$\mathsf{H}^i_{\mathsf{U}_{r,n}}(x) = \sum_{j=0}^{r-1} |W_{j,i}(\mathsf{U}_{r,n})| \; d_j(x) \cdot x^{r-j-1}.$$

Proof. This follows from [Bin26b, Proposition 4.8 and Theorem 4.15], which says that

$$|W_{j,i}(\mathsf{U}_{r,n})| = \sum_{\ell=1}^{n-r-i+1} \binom{n}{j}\binom{n-j-\ell}{r-j-1}\binom{n-r-\ell}{i-1},$$

and

$$\mathsf{H}^i_{\mathsf{U}_{r,n}}(x) = \sum_{j=0}^{r-1}\sum_{\ell=1}^{n-r-i+1} \binom{n}{j}\binom{n-j-\ell}{r-j-1}\binom{n-r-\ell}{i-1} d_j(x) \cdot x^{r-j-1}. \qquad \square$$

We note that, as $d_1(x) = 0$, $|W_{1,i}(\mathsf{U}_{r,n})|$ plays no role in the formula.

We now describe a method of counting and constructing elements of $W_{j,i}(\mathsf{U}_{r,n})$. We begin by constructing weakly retral chains $\mathscr{F}$ with $\operatorname{rks}(\mathscr{F}) = \{0, j, \ldots, r-1\}$.

Lemma 2.7. *Fix a uniform matroid $\mathsf{U}_{r,n}$ and let $0 \leq j \leq r-1$ with $j \neq 1$. Given a hyperplane H and a flat $G \leq H$ of rank j, there is a unique weakly retral chain $\mathscr{F}$ which contains G and H and has $\operatorname{rks}(\mathscr{F}) = \{0, j, \ldots, r-1\}$.*

Moreover, if H covers the flat F in $\mathscr{F}$, then $H \setminus F = \min(H \setminus G)$.

Proof. If $r = 1$ there is nothing to prove. When $r \geq 2$, write $H \setminus G = \{x_1 > \cdots > x_{r-j-1}\}$ and consider

$$\mathscr{F} = \{\emptyset < G < G \cup \{x_1\} < G \cup \{x_1, x_2\} < \cdots < G \cup \{x_1, \ldots, x_{r-j-2}\} < H\}.$$

If $j = 0$, then $G = \emptyset$, which we do not list twice in the chain. We claim that $\mathscr{F}$ is the unique weakly retral which contains G and H and has $\operatorname{rks}(\mathscr{F}) = \{0, j, \ldots, r-1\}$. Indeed, the cover condition for $G \cup \{x_1, \ldots, x_{k-1}\} < G \cup \{x_1, \ldots, x_k\}$ is equivalent to the condition that $x_k > x_{k+1}$. Therefore, a chain of flats

$$\mathscr{F}' = \left\{\emptyset < G < G \cup \{x'_1\} < \cdots < G \cup \{x'_1, \ldots, x'_{r-j-2}\} < G \cup \{x'_1, \ldots, x'_{r-j-1}\} = H\right\}$$

with $\operatorname{rks}(\mathscr{F}') = \{0, j, \ldots, r-1\}$ is weakly retral if and only if $H \setminus G = \left\{x'_1 > \cdots > x'_{r-j-1}\right\}$. (Note that the assumption $j > 1$ implies G does not cover $\emptyset$, so there is no restriction on G. Similarly, weakly retral chains do not impose any conditions on the hyperplane H.)

The last statement follows immediately. $\square$

We now extend this to a construction of pairs $(\mathscr{F}, \xi) \in W_{j,i}(\mathsf{U}_{r,n})$. For notation, given an ordered set $A = \{a_1 < \cdots < a_n\}$, we let $\min_i A = \{a_i\}$ and $\min_i^j A = \{a_i < \cdots < a_j\}$. For a set S and integer n, we let $\binom{S}{n}$ be the set of subsets of S with cardinality n.

Lemma 2.8. *Let $0 \leq j \leq r-1$, $j \neq 1$, and $1 \leq i \leq n-r$. Elements of $W_{j,i}(\mathsf{U}_{r,n})$ are in bijection with the following data:*

(1) An integer $0 \le k \le j$*,*
(2) an integer $0 \le m \le r-k-1$*,*
(3) a subset $A \in \binom{[n-m-1]}{r-m+i-1}$*,*
(4) a subset $B \in \binom{A \setminus \min_1^{k+1} A}{i-1}$*, and*
(5) a subset $C \in \binom{(A \cup \{n-m+1,\dots,n\}) \setminus (B \cup \min_1^{k+1} A)}{j-k}$*.*

Consequently,

$$|W_{j,i}(\mathsf{U}_{r,n})| = \sum_{k=0}^{j} \sum_{m=0}^{r-k-1} \binom{n-m-1}{r-m+i-1} \binom{r-m+i-1-(k+1)}{i-1} \binom{r-k-1}{j-k}.$$

Proof. Let D denote the set of data (k, m, A, B, C) satisfying conditions (1)–(5). We construct a map $\phi\colon W_{j,i}(\mathsf{U}_{r,n}) \to D$ which we show is a bijection.

Definition of ϕ: Fix $(\mathscr{F}, \xi) \in W_{j,i}(\mathsf{U}_{r,n})$. Let G be the rank j flat of $\mathscr{F}$, let H be the hyperplane of $\mathscr{F}$, and let $\xi = (\ell_{u_1} - \ell_{v_1}) \wedge \cdots \wedge (\ell_{u_i} - \ell_{v_i})$ as in Definition 2.3. We then define $\phi(\mathscr{F}, \xi) = (k, m, A, B, C)$, where

(1) k is the largest integer such that G contains $\min_1^k(H \cup \{u_1, \dots, u_i\})$,
(2) m is the largest integer such that H contains $\{n-m+1, \dots, n\}$,
(3) $A = (H \cup \{u_1, \dots, u_i\}) \setminus \{n-m+1, \dots, n\}$,
(4) $B = \{u_2, \dots, u_i\}$, and
(5) $C = G \setminus \min_1^k G$.

We now verify that $\phi(\mathscr{F}, \xi) \in D$.

(1) As $|G| = j$, we find that $0 \le k \le j$.
(2) Note that $\min_1^k(H \cup \{u_1, \dots, u_i\}) \cap \{n-m+1, \dots, n\} = \emptyset$, because
$$\min_k(H \cup \{u_1, \dots, u_i\}) < u_1 < n-m+1.$$
As $\min_1^k(H \cup \{u_1, \dots, u_i\}) \cup \{n-m+1, \dots, n\} \subseteq H$, we then find that $k+m \le r-1$, and $0 \le m \le r-k-1$.
(3) It is clear that $|A| = r-m+i-1$. We must show that $A \subseteq [n-m-1]$, and for this it suffices to show that $n-m \notin (H \cup \{u_1, \dots, u_i\})$. By the maximality of m, $n-m \notin H$. As $u_1 < \cdots < u_i < v_i < n-m+1$, we see that $n-m \notin \{u_1, \dots, u_i\}$ as well.
(4) Clearly $|B| = i-1$. As B is disjoint from H, we find that $B \subseteq A \setminus \min_1^k A$. To show that $\min_{k+1} A \notin B$, we will show that $u_1 = \min_{k+1} A$. By the maximality of k, either $\min_{k+1} A \in H \setminus G$ or $\min_{k+1} A = u_1$. If $\min_{k+1} A \in H \setminus G$, then $\min_{k+1} A = \min H \setminus G$. By Lemma 2.7, $u_1 < \min_{k+1} A$ which is impossible, as $u_1 \in A$ and $\min_k A < u_1$ as well. Therefore $u_1 = \min_{k+1} A$.
(5) It is clear that $|C| = j-k$. It remains to see that $C \subseteq (H \cup \{u_1, \dots, u_i\}) \setminus (B \cup \min_1^{k+1} A)$. This follows as $G \subseteq H$, and $B \cup \min_1^{k+1} A = \min_1^k G \cup \{u_1, \dots, u_i\}$.

Injectivity of ϕ: To see that ϕ is an injection, we show that we can recover G and H, the rank j and rank $r-1$ flats of $\mathscr{F}$, respectively, and $\{u_1, \dots, u_i\}$ from $\phi(\mathscr{F}, \xi) =$

(k, m, A, B, C). By Lemma 2.7, G and H determine $\mathscr{F}$, while $\{u_1, \dots, u_i\}$ determines ξ by the definition of admissible wedges. Above we showed that $u_1 = \min_{k+1} A$, so $\{u_1, \dots, u_i\} = B \cup \min_{k+1} A$. Then $H = A \cup \{n-m+1, \dots, n\} \setminus (B \cup \min_{k+1} A)$, and $G = C \cup \min_1^k A$.

Surjectivity of ϕ: Finally, we show that ϕ is surjective. Given $(k, m, A, B, C) \in D$, let $\{u_1 < \cdots < u_i\} = B \cup \min_{k+1} A$, $H = (A \cup \{n-m+1, \dots, n\}) \setminus \{u_1, \dots, u_i\}$, and $G = C \cup \min_1^k A$. By Lemma 2.7, there is a unique weakly retral chain $\mathscr{F}$ containing G and H with $\mathrm{rks}(\mathscr{F}) = \{0, j, \dots, r-1\}$. Define ξ to be the basic wedge

$$(\ell_{u_1} - \ell_{v_1}) \wedge \cdots \wedge (\ell_{u_i} - \ell_{v_i})$$

with v_k the successor of u_k in $[n] \setminus H$. We note that these successors exist because $n-m \notin H$, and $u_i < n-m$.

We begin by showing that $(\mathscr{F}, \xi) \in W_{j,i}(\mathsf{U}_{r,n})$, and here it suffices to show that ξ is admissible. The only non-trivial condition to check is that, if H covers the flat F, then $u_1 < H \setminus F$. From Lemma 2.7, $H \setminus F = \min(H \setminus G)$. As $u_1 = \min_{k+1} A$, and $\min_1^k A \subseteq G$, we find that $u_1 < H \setminus F$ as desired.

We conclude by verifying that $\phi(\mathscr{F}, \xi) = (k, m, A, B, C)$. For notation, we will write $\phi(\mathscr{F}, \xi) = (k', m', A', B', C')$.

(1) As $\min_1^k A \subseteq G$ and $\min_{k+1} A = u_1 \notin G$, we see that k is the largest integer such that $\min_1^k(H \cup \{u_1, \dots, u_i\}) \subseteq G$, which is the definition of k'.
(2) As $A \subseteq [n-m-1]$, we find that $n-m \notin H$, so $m' \leq m$. On the other hand, $\{n-m+1, \dots, n\} \subseteq H$, so $m' \geq m$. Thus $m' = m$.
(3) By definition, $A' = (H \cup \{u_1, \dots, u_i\}) \setminus \{n-m+1, \dots, n\}$. As

$$H = (A \cup \{n-m+1, \dots, n\}) \setminus \{u_1, \dots, u_i\},$$

we see that $A' = A$.
(4) By definition, $B' = \{u_2, \dots, u_i\}$. On the other hand, $B = \{u_1, \dots, u_i\} \setminus \min_{k+1} A$. As $\min_{k+1} A = u_1$, $B' = B$.
(5) By definition $C' = G \setminus \min_1^k G$. As $G = C \cup \min_1^k A$ and $\min_1^k G = \min_1^k A$, we see that $C' = C$.

□

3. Background on real-rootedness

Let $f(x) = a_0 + a_1 x + \cdots + a_d x^d$. We say that $f(x)$ is *real-rooted* if all of its complex zeros are real. A finite sequence of nonnegative real numbers $(a_0, a_1, \dots, a_d)$ is called *log-concave* if $a_k^2 \geq a_{k-1} a_{k+1}$ for every $1 \leq k \leq d-1$. It is called *unimodal* if there is an index $0 \leq m \leq d$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_d.$$

We say that a polynomial is log-concave or unimodal if its coefficient sequence has the corresponding property. The following implications are standard.

Proposition 3.1. *Let $f(x)$ be a polynomial with nonnegative coefficients and no internal zeros. Then*

$$f(x) \text{ is real-rooted} \implies f(x) \text{ is log-concave} \implies f(x) \text{ is unimodal.}$$

Interlacing gives a useful way to compare the zeros of two real-rooted polynomials. Let $f(x)$ and $g(x)$ be real-rooted polynomials with positive leading coefficients. Write the zeros of f and g, counted with multiplicity, as $\cdots \leq \alpha_2 \leq \alpha_1$ and $\cdots \leq \beta_2 \leq \beta_1$, respectively. We say that g *weakly interlaces* f if

$$\cdots \leq \beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1.$$

We use the notation $g \prec f$ to mean that the zeros of g weakly interlace those of f.

3.1. **The deranged map and the magic basis.** Let $\mathfrak{S}_n$ denote the symmetric group on n letters and let $\mathfrak{D}_n \subseteq \mathfrak{S}_n$ denote the set of derangements. For $\omega \in \mathfrak{S}_n$, let $\operatorname{exc}(\omega) = \big|\{k \mid \omega(k) > k\}\big|$ be the number of excedances of ω. The Eulerian and derangement polynomials are

$$A_n(x) = \sum_{\omega\in\mathfrak{S}_n} x^{\operatorname{exc}(\omega)} \qquad d_n(x) = \sum_{\omega\in\mathfrak{D}_n} x^{\operatorname{exc}(\omega)}.$$

By convention, we define $d_0(x) = 1$.

Brändén and Solus study the zeros of Eulerian and derangement polynomials and colored derangement polynomials by defining the following operator [BS21, Section 3.2]. The *deranged map* is the unique linear operator $\mathcal{D} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ satisfying

$$\mathcal{D}(x^n) = d_n(x)$$

for every $n \geq 0$.

For a fixed nonnegative integer r, consider the basis $\mathcal{B}_r = \left\{x^k(1+x)^{r-k} : 0 \leq k \leq r\right\}$ of the vector space of polynomials of degree at most r. We refer to $\mathcal{B}_r$ as the *magic basis*. The following result of Brändén and Solus is the main real-rootedness criterion that we will use.

Theorem 3.2 ([BS21, Corollary 3.7])**.** *Let $f(x) = \sum_{k=0}^{r} h_k x^k(1+x)^{r-k}$, with $h_k \geq 0$. Then $A_r(x) \prec \mathcal{D}(f) \prec d_r(x)$. In particular, $\mathcal{D}(f)$ is real-rooted.*

A generalization of this map has been recently introduced by Brändén and the second author to provide real-rootedness results for the Chow polynomials of arbitrary posets, a generalization of $\mathsf{H}^0_M(x)$ from geometric lattices to arbitrary finite graded bounded posets [BV25].

4. Refined Hodge–Poincaré polynomials of uniform matroids are real-rooted

In this section we prove Theorem 1.2. To do so, we consider the involution $\mathcal{I}_r$ defined on polynomials of degree at most r by

$$\mathcal{I}_r(f(x)) = x^r f(x^{-1}).$$

Clearly, if f is a polynomial of degree at most r, then f is real-rooted if and only if $\mathcal{I}_r(f)$ is, since the zeros of f are the reciprocals of the zeros of $\mathcal{I}_r(f)$.

Theorem 4.1. *For every $1 \le r \le n$ and every $0 \le i \le n-r$, the refined Hodge–Poincaré polynomial $\mathsf{H}^i_{\mathsf{U}_{r,n}}(x)$ has only real zeros.*

Proof. Fix a refined Hodge–Poincaré polynomial $\mathsf{H}^i(\mathsf{U}_{r,n})(x)$. As the case $i=0$ produces the Chow polynomial, we may assume that $i \ge 1$.

Define the polynomial

$$f(x) = \sum_{k=0}^{r-1}\sum_{m=0}^{k} \binom{n-m-1}{r+i-m-1}\binom{i+k-m-1}{i-1} x^{r-1-k}(1+x)^k.$$

By Theorem 3.2, $\mathcal{D}(f)$ is real-rooted. Therefore $\mathcal{I}_{r-1} \circ \mathcal{D}(f)$ is real-rooted as well. We complete the proof by showing that $\mathsf{H}^i(\mathsf{U}_{r,n})(x) = \mathcal{I}_{r-1} \circ \mathcal{D}(f)$.

Expanding $f(x)$,

$$\begin{aligned}
f(x) &= \sum_{k=0}^{r-1}\sum_{m=0}^{k} \binom{n-m-1}{r+i-m-1}\binom{i+k-m-1}{i-1} x^{r-1-k}(1+x)^k \\
&= \sum_{k=0}^{r-1}\sum_{j=0}^{k}\sum_{m=0}^{k} \binom{n-m-1}{r-m+i-1}\binom{i+k-m-1}{i-1}\binom{k}{j} x^{r-1-k+j} \\
&= \sum_{k=0}^{r-1}\sum_{j=0}^{r-k-1}\sum_{m=0}^{r-k-1} \binom{n-m-1}{r-m+i-1}\binom{i+r-k-1-m-1}{i-1}\binom{r-k-1}{j} x^{k+j} \\
&= \sum_{k=0}^{r-1}\sum_{j=k}^{r-1}\sum_{m=0}^{r-k-1} \binom{n-m-1}{r-m+i-1}\binom{i+r-k-1-m-1}{i-1}\binom{r-k-1}{j-k} x^{j} \\
&= \sum_{j=0}^{r-1}\sum_{k=0}^{j}\sum_{m=0}^{r-k-1} \binom{n-m-1}{r-m+i-1}\binom{i+r-k-1-m-1}{i-1}\binom{r-k-1}{j-k} x^{j} \\
&= \sum_{j=0}^{r-1} \left|W_{j,i}(\mathsf{U}_{r,n})\right| x^j.
\end{aligned}$$

Applying the deranged map $\mathcal{D}$ and involution $\mathcal{I}_{r-1}$,

$$\begin{aligned}
\mathcal{D}(f) &= \sum_{j=0}^{r-1} \left|W_{j,i}(\mathsf{U}_{r,n})\right| d_j(x) \\
\mathcal{I}_{r-1} \circ \mathcal{D}(f) &= \sum_{j=0}^{r-1} \left|W_{j,i}(\mathsf{U}_{r,n})\right| d_j(x) \cdot x^{r-j-1},
\end{aligned}$$

where in the last equality we use the fact that the jth derangement polynomial is palindromic with center of symmetry $j/2$, i.e., $\mathcal{I}_j(d_j(x)) = d_j(x)$ and thus $\mathcal{I}_{r-1}(d_j(x)) = d_j(x)x^{r-1-j}$. Therefore $\mathsf{H}^i(\mathsf{U}_{r,n})(x) = \mathcal{I}_{r-1} \circ \mathcal{D}(f)$ by Lemma 2.6. $\square$

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