TOR GROUPS OF THE STANLEY–REISNER RING OF A MATROID

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Abstract. We introduce the Tor groups $\mathrm{Tor}_{\bullet}^{S^{\circ}_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M}}],\mathbb{C})_{\bullet}$ for a loopless matroid M as a way to study the extra relations occurring in the linear ideal of the Feichtner–Yuzvinski presentation of the Chow ring $A^{\bullet}(\mathrm{M})$. This extends the definition of the Chow ring of a matroid since $\mathrm{Tor}_{0}^{S^{\circ}_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M}}],\mathbb{C})_{\bullet} \cong A^{\bullet}(\mathrm{M})$. We show that these Tor groups fit into a long exact sequence arising from the matroidal flips of Adiprasito, Huh, and Katz [1], extending the short exact sequence in the case of Chow rings. Using this long exact sequence we give a recursive formula for the Hilbert series of the Tor algebra of a uniform matroid.

1. Introduction

The Chow rings of matroids have drawn considerable attention in the past decade. For a loopless matroid M, the Chow ring $A^{\bullet}(\mathrm{M})$ acts like the cohomology ring of a projective variety [1]. In particular, there is a well-defined *intersection theory*, and the intersection numbers yield combinatorial data, such as the *catenary data* [11], of the matroid. Whenever M is representable (say over $\mathbb{C}$) as a hyperplane arrangement, the Chow ring $A^{\bullet}(\mathrm{M})$ is defined to be the Chow ring of a smooth projective variety. Namely, one takes a wonderful compactification in the sense of de Concini–Processi [7]. In order to extend the definition to the non-representable case, Feichtner and Yuzvisnky [10] construct a different geometric model for a loopless matroid M, whether representable or not. This model is the toric variety $X(\Sigma_{\mathrm{M}})$ (defined over $\mathbb{C}$) associated to the *Bergman fan* Σ_{M}. Whenever M is representable, the Chow ring $A^{\bullet}(\Sigma_{\mathrm{M}})$ is isomorphic to $A^{\bullet}(\mathrm{M})$, so this defines $A^{\bullet}(\mathrm{M})$ for all loopless matroids.

In this paper, we will both investigate finer algebraic structure in Feichtner and Yuzvinsky's construction of the Chow ring and consider the full cohomology ring of $X(\Sigma_{\mathrm{M}})$, geometrically extending the Chow ring of the matroid. It is perhaps surprising that these two directions, algebraic and geometric, are related. Precisely, they are both answered by the Tor groups of the Stanley–Reisner ring of the matroid. This interplay between the algebraic structure of the Feichtner–Yuzvinsky presentation of $A^{\bullet}(\mathrm{M})$ and the geometry of $X(\Sigma_{\mathrm{M}})$ is the main theme of the paper, and we obtain our main results by passing between these algebraic and geometric perspectives.

One can view this approach as extending the perspective of Hochster's Formula, which computes the Tor groups of the Stanley–Reisner ring of a simplicial complex (over the polynomial ring generated by its vertices) in terms the reduced homology of the simplicial complex. In our case, we use the correspondence due to Matthias Franz [13, 14] between the Tor groups of the Stanley–Reisner ring of a (smooth) simplicial fan and the cohomology of the toric variety associated to the fan. The cohomology of a toric variety depends not only on the combinatorial type of the fan but also on its embedding in an integral lattice; this is reflected in the ring we take Tor over, as well as the algebra structure we place on the Stanley–Reisner ring.

Let us begin by defining the finer algebraic structure we consider in Feichtner and Yuzvinsky's construction of the Chow ring of a matroid. As this is the Chow ring of a toric variety, it has an explicit presentation in terms of a Stanley–Reisner ring modulo a linear ideal. Define the *Stanley–Reisner ring*[1] of the fan Σ_{M} to be

$$\mathbb{C}[\Sigma_{\mathrm{M}}] = \frac{\mathbb{C}[x_F : F \text{ is a proper, non-empty flat of M}]}{\langle x_F x_G : F \text{ and } G \text{ are incomparable}\rangle.}$$

Then

$$A^\bullet(\mathrm{M}) \cong \frac{\mathbb{C}[\Sigma_{\mathrm{M}}]}{\langle \sum_{\{F: i\in F\}} x_F - \sum_{\{G: j\in G\}} x_G\rangle_{i,j\in \mathrm{M}}.} \tag{1}$$

We are interested in the relations between the generators of this ideal and what such relations tell us about M. By counting dimension, one expects the generators $\left\{\sum_{\{F:i\in F\}} x_F - \sum_{\{G:n\in G\}} x_G\right\}_{i=1}^{n-1}$ of the ideal to not be $\mathbb{C}[\Sigma_{\mathrm{M}}]$-regular for a general matroid M on the ground set $[n] = \{1, \dots, n\}$. That is, there should be further relations. Take for example the uniform matroid $\mathbb{U}_{3,5}$. There are 15 monomials of degree 1 and 35 monomials of degree 2 in $\mathbb{C}[\Sigma_{\mathbb{U}_{3,5}}]$, while there are 4 linearly independent generators of the ideal of relations, each of degree 1. By Poincaré duality, $\dim A^0(\mathbb{U}_{3,5}) = \dim A^2(\mathbb{U}_{3,5}) = 1$, so the dimension of the space of relations is at least 26. The following algebraic question naturally arises: For which matroids are the generators of the linear ideal in Equation (1) $\mathbb{C}[\Sigma_{\mathrm{M}}]$-regular, and if this is not the case, what combinatorial information about the matroid do these extra relations capture?

We measure these relations through Tor groups. For a matroid M on the ground set $[n]$, define the rings

$$S_{\mathrm{M}} = \mathbb{C}[x_1, \dots, x_n] \text{ and } S_{\mathrm{M}}^\circ = \mathbb{C}[x_1 - x_n, \dots, x_{n-1} - x_n] \subseteq S_{\mathrm{M}}.$$

The map

$$S_{\mathrm{M}} \longrightarrow \mathbb{C}[\Sigma_{\mathrm{M}}], \quad x_i \longmapsto \sum_{i\in F} x_F$$

endows $\mathbb{C}[\Sigma_{\mathrm{M}}]$ with the structure of an S_{M}-algebra and, by precomposing with the inclusion $S_{\mathrm{M}}^\circ \hookrightarrow S_{\mathrm{M}}$, an S_{M}°-algebra. The cokernel of the map $S_{\mathrm{M}}^\circ \to \mathbb{C}[\Sigma_{\mathrm{M}}]$ is precisely $A^\bullet(\mathrm{M}) \cong \mathbb{C}[\Sigma_{\mathrm{M}}] \otimes_{S_{\mathrm{M}}^\circ} \mathbb{C}$, where $\mathbb{C}$ is an S_{M}°- and S_{M}-algebra under the identification $\mathbb{C} \cong S_{\mathrm{M}}/\langle x_1, \dots, x_n\rangle$. Therefore the relations between generators in the linear ideal defining the Chow ring is measured by $\mathrm{Tor}_\bullet^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M}}], \mathbb{C})_\bullet$, and the Chow ring itself is $\mathrm{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M}}], \mathbb{C})_\bullet$. This Tor algebra naturally carries a bigrading $\mathrm{Tor}_\bullet^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M}}], \mathbb{C})_\bullet = \bigoplus_{s,t} \mathrm{Tor}_t^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M}}], \mathbb{C})_s$. The grading t, or *Tor degree*, comes from the indexing of the Tor functors, while the grading s, or *Stanley–Reisner degree*, is inherited from the grading of $\mathbb{C}[\Sigma_{\mathrm{M}}]$ as a polynomial ring.

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[1] This is isomorphic to the *face ring* or *Stanley–Reisner ring*, in the sense of Stanley [23, Chapter 2], of the order complex of the lattice of flats of M.

On the geometric side, we study the cohomology of the toric varieties $X(\Sigma_{\mathrm{M}})$. In contrast to the wonderful compactifications of hyperplane arrangement complements, the varieties $X(\Sigma_{\mathrm{M}})$ are not complete, except when M is a boolean matroid. The cohomology $H^\bullet(X(\Sigma_{\mathrm{M}});\mathbb{C})$ carries a Hodge–Deligne filtration, and the non-completeness suggests there is non-zero cohomology outside of type (p,p). Therefore the Chow ring, which is purely of type (p,p), does not see this cohomology. One goal is to compute the cohomology ring $H^\bullet(X(\Sigma_{\mathrm{M}});\mathbb{C})$ in terms of the combinatorial structure of M.

This cohomology ring is also isomorphic to $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M}}],\mathbb{C})_\bullet$. The open, dense torus T_{M} acts upon the toric variety $X(\Sigma_{\mathrm{M}})$, and with this action, the equivarient cohomology is $H^\bullet_{T_{\mathrm{M}}}(X(\Sigma_{\mathrm{M}});\mathbb{C}) \cong \mathbb{C}[\Sigma_{\mathrm{M}}]$ (see [4]), while $H^\bullet_{T_{\mathrm{M}}}(\mathrm{pt};\mathbb{C}) \cong S^\circ_{\mathrm{M}}$. The key result, due to Franz [13], is that $X(\Sigma_M)$ is *formal*, in that as modules

$$H^\bullet(X(\Sigma_{\mathrm{M}});\mathbb{C}) \cong H^\bullet(H^\bullet_{T_{\mathrm{M}}}(X(\Sigma_{\mathrm{M}});\mathbb{C}) \otimes_{S^\circ_{\mathrm{M}}} \bigwedge L),$$

where $L = H^{>0}_{T_{\mathrm{M}}}(\mathrm{pt};\mathbb{C})$, and the right hand side is the cohomology of the Koszul complex $H^\bullet_{T_{\mathrm{M}}}(X(\Sigma_{\mathrm{M}});\mathbb{C}) \otimes \bigwedge^\bullet L$. As $\bigwedge^\bullet L$ is a free resolution of $\mathbb{C}$ in the category of S°_{M}-modules,

$$H^\bullet(X(\Sigma_{\mathrm{M}});\mathbb{C}) \cong \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M}}],\mathbb{C})_\bullet.$$

Moreover, the Tor algebra remembers the mixed Hodge structure of $H^\bullet(X(\Sigma_{\mathrm{M}});\mathbb{C})$ (see [24] for details) and the product operation in the Tor algebra, induced by multiplication in the Koszul complex, is precisely the cup product in cohomology [14].

One of the main geometric tools we use in studying $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M}}],\mathbb{C})_\bullet$ are the *matroidal flips* introduced in [1]. Each flip is a tropical modification of fans $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$, and on toric varieties, this corresponds to blowing up $X(\Sigma_{\mathrm{M},\mathcal{P}_-})$ along a subvariety and then removing some subvarieties. Together these matroidal flips interpolate between the fans $\Sigma_{\mathrm{M},\emptyset}$ and Σ_{M}.

Just as with the Bergman fan Σ_{M}, each fan $\Sigma_{\mathrm{M},\mathcal{P}}$ has an associated Stanley–Reisner ring $\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]$ which is an S_{M}- and S°_{M}-algebra. Therefore we can also consider the Tor groups $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}],\mathbb{C})_\bullet$ and the Chow rings

$$A^\bullet(\mathrm{M},\mathcal{P}) := A^\bullet(X(\Sigma_{\mathrm{M},\mathcal{P}})) \cong \mathrm{Tor}_0^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}],\mathbb{C})_\bullet.$$

In [1, Theorem 6.18], the authors show matroidal flips induce a short exact sequence of Chow rings[2]

$$0 \to A^\bullet(\mathrm{M},\mathcal{P}_-) \to A^\bullet(\mathrm{M},\mathcal{P}_+) \to A^{>0}(\mathrm{M}^Z,\emptyset) \otimes A^\bullet(\mathrm{M}^Z) \to 0$$

where $Z = \mathcal{P}_+ \setminus \mathcal{P}_-$. By viewing the Tor groups as cohomology rings of the corresponding toric varieties and matroidal flips as close to being a blow up, we show this short exact sequence of Chow rings extends to a long exact sequence of Tor groups involving $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}],\mathbb{C})_\bullet$ and $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}],\mathbb{C})_\bullet$ (Theorem 5.7 and Corollary 6.3). From this exact sequence we deduce properties of $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M}}],\mathbb{C})_\bullet$ from those of $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}],\mathbb{C})_\bullet$.

[2] In the language of [1], $A^\bullet(\mathrm{M},\mathcal{P}_-)$ is the image of the pull-back homomorphism Φ_Z, and $A^{>0}(\mathrm{M}^Z,\emptyset) \otimes A^\bullet(\mathrm{M}^Z)$ is the image of the Gysin homomorphisms $\oplus_1^{\mathrm{rank}(Z)-1}\Psi_Z^{p,q}$.

1.1. **Main Results.** The first of our two main results is a sharp vanishing theorem for the Tor algebra based on the combinatorics of M.

Theorem 6.1. Let M be a loopless matroid of rank r on the ground set $[n]$. For $t > n - r$ or $s > r - 1$, $\mathrm{Tor}_t^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_M], \mathbb{C})_s = 0$.

Moreover in top degree

$$\dim \mathrm{Tor}_{n-r}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_M], \mathbb{C})_{r-1} = |\mathrm{nbc}(\mathrm{M})| \neq 0.$$

Here nbc(M) denotes the set of no-broken-circuit bases of M; the number |nbc(M)| is also known as the *Möbius invariant of* M, denoted $\widetilde{\mu}(\mathrm{M})$ ([2, pg. 241]).

As an immediate corollary, we characterize the loopless matroids M for which the sequence $\left\{\sum_{\{F:i\in F\}} x_F - \sum_{\{G:n\in G\}} x_G\right\}_{i=1}^{n-1}$ is $\mathbb{C}[\Sigma_{\mathrm{M}}]$-regular.

Corollary 1.1. *The sequence* $\left\{\sum_{\{F:i\in F\}} x_F - \sum_{\{G:n\in G\}} x_G\right\}_{i=1}^{n-1}$ *is* $\mathbb{C}[\Sigma_{\mathrm{M}}]$*-regular precisely when* M *is a boolean matroid.*

As $\mathrm{Tor}_\bullet^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}], \mathbb{C})_\bullet$ is bigraded, there is a *bigraded Hilbert series*

$$\mathrm{Hilb}(\mathrm{Tor}_\bullet^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}], \mathbb{C})_\bullet) = \sum_{i,j} \dim \mathrm{Tor}_i^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}], \mathbb{C})_j\, x^i y^j.$$

The second of our main results gives a recursive formula for the Hilbert series of the Tor algebra of a uniform matroid.

Theorem 6.5. Let r and k be positive integers. Then

$$\begin{aligned}\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathbb{U}_{r,k+r}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathbb{U}_{r,k+r}}], \mathbb{C}\right)_\bullet\right) &= \mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathbb{U}_{r,k+r}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathbb{U}_{r,k+r},\emptyset}], \mathbb{C}\right)_\bullet\right)\\ &+ \sum_{i=1}^{r-1}\binom{r+k}{i}\left(\frac{y-y^i}{1-y}\right)\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathbb{U}_{r-i,r+k-i}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathbb{U}_{r-i,r+k-i}}], \mathbb{C}\right)_\bullet\right).\end{aligned}$$

1.2. **Outline.** Let us briefly outline the structure of this paper. In Section 2 we recall the basic notations from matroid theory and commutative algebra which we use. Some small lemmas are also proved in this section. In Sections 3 and 4 we compute the Hilbert series of $\mathrm{Tor}_\bullet^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}], \mathbb{C})_\bullet$ and $\mathrm{Tor}_\bullet^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}], \mathbb{C})_\bullet$, respectively; this latter group plays the role of the base case of the matroidal flip induction. In Section 5 we introduce the machinery of matroidal flips and show they induce a long exact sequence of Tor groups. Finally, in Section 6 we prove the announced results using these long exact sequences.

We note the majority of the content of Section 3 appears in previous literature. The algebra $\mathrm{Tor}_\bullet^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}], \mathbb{C})_\bullet$ has been studied in [9, 18] in relation to *facet ideals*. As far as we are aware, the explicit computation we produce of $\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}], \mathbb{C})_\bullet\right)$ as an evaluation of the Tutte polynomial of M has not yet appeared; however, the Hilbert series $\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathrm{M}}}(\mathbb{C}[\overline{\Sigma_{\mathrm{M},\emptyset}}], \mathbb{C})_\bullet\right)$ for the Alexander dual of $\Sigma_{\mathrm{M},\emptyset}$ has been written down in [12, 19, 20], and the authors of [19] use the Betti numbers of the Stanley–Reisner ring of the Alexander dual to write an equation for the evaluation of the Tutte polynomial we produce. We refer the reader to Remark 3.10 for further discussion.

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2. Notation

2.1. **Matroids.** We begin by reviewing the basic definitions and terminology of matroids. We recommend [22] or [26] for more details.

2.1.1. *Definitions of a Matroid.* Let E be a finite set. We define a *matroid* M *on* E to be a non-empty collection $\mathcal{B}(\mathrm{M})$ of subsets of E which satisfies the following exchange property: For $B_1, B_2 \in \mathcal{B}(\mathrm{M})$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}(\mathrm{M})$. We call elements of $\mathcal{B}(\mathrm{M})$ *bases*.

A subset of E is *independent* if it is contained in some basis of M; otherwise the set is *dependent*. A *circuit* is a minimal dependent subset, and a *loop* is a circuit of one element.

Independent sets give rise to the *rank function* of M,

$$\mathrm{rank}_{\mathrm{M}} \colon 2^{[n]} \to \mathbb{Z}_{\geq 0}, \quad \mathrm{rank}_{\mathrm{M}}(A) = \max\left\{|I| : I \subseteq A \text{ is independent}\right\}.$$

A *flat of rank* j is a maximal element in the set of rank j subsets of $[n]$. The set of all flats of M forms a lattice which we denote $\mathcal{L}(\mathrm{M})$. If we remove the minimal and maximal elements, say $\widehat{0}$ and $\widehat{1}$, of $\mathcal{L}(\mathrm{M})$, we create the partially ordered set of proper non-empty flats of M. We denote this by $\widehat{\mathcal{L}}(\mathrm{M})$. We say I *spans* a flat F if $I \subseteq F$ and $\mathrm{rank}_{\mathrm{M}}(I) = \mathrm{rank}_{\mathrm{M}}(F)$.

Unless otherwise stated, our convention will be that a matroid M is defined on the ground set $[n] = \{1, \ldots, n\}$ and has rank $r > 0$.

2.1.2. *Operations on Matroids.* We now define three operations on matroids: restriction, contraction, and duality.

First we define *matroid restriction.* For a flat F, the matroid M^F is defined on the ground set F with bases

$$\mathcal{B}\left(\mathrm{M}^F\right) = \{I \subseteq F : \ I \text{ is an independent set of M and } \mathrm{rank}_{\mathrm{M}}(I) = \mathrm{rank}_{\mathrm{M}}(F)\}.$$

The lattice of flats of $\mathcal{L}(\mathrm{M}^F)$ is identified with the interval $[\hat{0}, F]$ in $\mathcal{L}(\mathrm{M})$. This definition extends verbatim for the restriction to any subset $W \subseteq [n]$, not just to flats. In this case it is customary to write the restriction as $\mathrm{M}|_W$.

Second we define *matroid contraction.* For a flat F, the matroid M_F is defined on the ground set $[n] \setminus F$ with bases

$$\mathcal{B}\left(\mathrm{M}_F\right) = \{I \subseteq [n] \setminus F : |I| = \mathrm{rank}_{\mathrm{M}}(\mathrm{M}) - \mathrm{rank}_{\mathrm{M}}(F) \text{ and } I \cup F \text{ spans M}\}.$$

Now the lattice of flats of $\mathcal{L}\left(\mathrm{M}_F\right)$ is identified, via union with F, the interval $[F, \hat{1}]$ in $\mathcal{L}(\mathrm{M})$.

Finally, the *dual matroid* M^* is defined on $[n]$ by

$$\mathcal{B}(\mathrm{M}^*) = \{[n] \setminus B : B \in \mathcal{B}(\mathrm{M})\}.$$

2.1.3. *Activity and Passivity in Matroids.* Our convention that matroids have ground set $[n]$ yields a preferred ordering on the elements of the ground set. This choice allows us to define the important notions of *activity* and *passivity* in matroids; all the results in this paper are independent of this choice of ordering.

Suppose B is a basis of M, and take $x \in [n] \setminus B$. Then $B \cup x$ is a dependent set, so it contains a unique circuit $C_{B\cup x}$ which we call the *fundamental circuit of* $B \cup x$. For any $y \in C_{B\cup x}$, the set $(B \cup x) \setminus y$ is a basis.

With respect to $B \in \mathcal{B}(\mathrm{M})$, an element $x \in [n] \setminus B$ is *externally active* if it is the smallest element in $C_{B\cup x}$. Otherwise, we say x is *externally passive*.

Using duality we have a similar notion of internal activity. With respect to a basis $B \in \mathcal{B}(\mathrm{M})$, an element $x \in B$ is *internally active* if it is externally active with respect to $\mathrm{M} \setminus B \in \mathcal{B}(\mathrm{M}^*)$.

Definition 2.1. For a basis B, we define $\mathrm{ea}(B)$, $\mathrm{ep}(B)$, and $\mathrm{ia}(B)$ to be the number of externally active, externally passive, and internally active elements with respect to B. Let $\mathrm{EP}(B)$ be the set of externally passive elements with respect to B.

The generating function of bases of M with given internal and external activity is the *Tutte polynomial*. This polynomial specializes to many classical invariants of graphs and matroids [5].

Definition 2.2. The *Tutte polynomial* of a matroid M is

$$T_{\mathrm{M}}(x, y) = \sum_{B \in \mathcal{B}(\mathrm{M})} x^{\mathrm{ia}(B)} y^{\mathrm{ea}(B)}.$$

Of special interest to us are the bases with no externally active elements, or equivalently, bases with the maximal number of externally passive elements. These are called *no-broken-circuit-*, or *nbc-*, bases. We will denote the set of nbc-bases of M by nbc(M).

We conclude this subsection by proving two lemmas on external passivity.

Lemma 2.3. *Let* M *be a loopless matroid on* $[n]$ *and* B *a basis. Then*

$$\binom{\mathrm{ep}(B)}{i} = |\{W \subseteq [n] : |W| = |B| + i \text{ and } B \in \mathrm{nbc}(\mathrm{M}|_W)\}|.$$

Proof. A subset $W \subseteq [n]$ satisfies $B \in \mathrm{nbc}(\mathrm{M}|_W)$ if and only if $B \subseteq W$ and every $x \in W \setminus B$ is externally passive with respect to B. So for a fixed i, the set on the right-hand-side of the statement is uniquely determined by choosing i elements of $\mathrm{EP}(B)$, and this is counted by the left-hand-side. □

The set $\mathcal{B}(\mathrm{M})$ is linearly ordered by the lexicographic order. Let $B_{\max}$ be the maximal element in this order.

Lemma 2.4. *Let* M *be a loopless matroid on* $[n]$. *Then* $\mathrm{ep}(B) = 0$ *if and only if* $B = B_{\max}$.

Proof. We prove the contrapositive. Suppose $\mathrm{ep}(B) \neq 0$; we will construct a basis which is greater than B lexicographically. Choose $x \in \mathrm{EP}(B)$ and $y \in C_{B\cup x}$ with $y < x$. The basis $(B \setminus y) \cup x$ is greater than B, hence $B \neq B_{\max}$.

Conversely, suppose $B \neq B_{\max}$, and let x be the maximal element in $B_{\max} \setminus B$. We want to show that $x \in \mathrm{EP}(B)$. To start, take $y \in C_{B\cup x}$ with $y \notin B_{\max}$, and it will suffice to show $y < x$. Consider the cycle $C_{B_{\max}\cup y}$. There exists $z \in C_{B_{\max}\cup y}$ with $z \notin B$. The maximality assumption on x implies $z < x$. Similarly, the fact $(B_{\max} \setminus z) \cup y$ is a basis, combined with the maximality of $B_{\max}$, implies $y < z$. Therefore we have shown $y < x$, so $x \in \mathrm{EP}(B)$, and $\mathrm{ep}(B) \neq 0$. □

2.2. **Bergman Fans and Stanley–Reisner Rings.** Let M be a loopless matroid on the ground set $[n]$. The main geometric objects we consider are the Bergman fans $\Sigma_{\mathrm{M},\emptyset}$ and Σ_{M} and their associated toric varieties $X(\Sigma_{\mathrm{M},\emptyset})$ and $X(\Sigma_{\mathrm{M}})$. We define these fans in the more general context of *Bergman fans associated to order filters*, following [1]. This generality is necessary in Section 5 to discuss matroidal flips.

Definition 2.5. An *order filter* $\mathcal{P}$ *of* M is a subset of $\widehat{\mathcal{L}}(\mathrm{M})$ which is closed under upward inclusion: If $F \leq G$ in $\widehat{\mathcal{L}}(\mathrm{M})$ and $F \in \mathcal{P}$, then $G \in \mathcal{P}$ as well.

The Bergman fans we consider live in the quotient space $\mathbb{R}^n/\mathbb{R}\cdot(1,\ldots,1)$. For coordinates, let $\{f_1,\ldots,f_n\}$ be a basis of $\mathbb{R}^n$, and consider the quotient $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}\cdot(1,\ldots,1)$. The elements $\{e_i := \pi(f_i)\}$ span an integral lattice N_{M}, and for a subset $S \subseteq [n]$ we define

$$e_S = \sum_{i\in S} e_i.$$

For a chain $\mathcal{F} \subseteq \widehat{\mathcal{L}}(\mathrm{M})$ and $I \subseteq [n]$, we write $I < \mathcal{F}$ if and only if I is contained in each element of $\mathcal{F}$. For $I < \mathcal{F}$ we define the cone

$$\sigma_{I<\mathcal{F}} = \mathrm{cone}\left(\{e_i, e_F : i \in I, F \in \mathcal{F}\}\right).$$

Definition 2.6. Let $\mathcal{P}$ be an order filter. The *Bergman fan* $\Sigma_{\mathrm{M},\mathcal{P}}$ is the integral fan in $\mathbb{R}^n/\mathbb{R}\cdot(1,\ldots,1)$ defined by

$$\Sigma_{\mathrm{M},\mathcal{P}} = \left\{\sigma_{I<\mathcal{F}} : I \text{ spans no element of } \mathcal{P}\cup\widehat{1} \text{ and } \mathcal{F}\subseteq\mathcal{P}\right\}.$$

In the case $\mathcal{P} = \widehat{\mathcal{L}}(\mathrm{M})$, we write Σ_{M} instead of $\Sigma_{\mathrm{M},\mathcal{P}}$.

Attached to a fan Σ is its *Stanley–Reisner ring* $\mathbb{C}[\Sigma]$ which captures the combinatorial stucture of Σ.

Definition 2.7. For a fan Σ, let $\Sigma(1)$ denote the set of all rays. The *Stanley–Reisner ring* of Σ is defined by

$$\mathbb{C}[\Sigma] = \frac{\mathbb{C}[x_\rho : \rho\in\Sigma(1)]}{I_\Sigma}, \quad I_\Sigma = \langle x_{\rho_1}x_{\rho_2}\cdots x_{\rho_s} : \mathrm{cone}\,(\rho_1,\rho_2,\ldots,\rho_s)\notin\Sigma\rangle.$$

The ideal I_Σ is called the *Stanley–Reisner ideal.*

For the fans $\Sigma_{\mathrm{M},\mathcal{P}}$ we will write x_i and x_F, instead of x_{e_i} and x_{e_F}, for the indeterminants corresponding to the rays e_i and e_F.

Define the rings

$$S_{\mathrm{M}} = \mathbb{C}[x_1, \ldots, x_n] \text{ and } S_{\mathrm{M}}^{\circ} = \mathbb{C}[x_1 - x_n, \ldots, x_{n-1} - x_n] \subseteq S_{\mathrm{M}}.$$

Note that these rings depend only of the ground set $[n]$ and not the bases of M. For $i \in [n]$ and Stanley–Reisner ring $\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]$, define the element

$$\delta_i x_i = \begin{cases} x_i & \text{if } x_i \in \mathbb{C}[\Sigma_{M,\mathcal{P}}] \\ 0 & \text{otherwise.} \end{cases}$$

The map

$$\begin{aligned} S_{\mathrm{M}} &\longrightarrow \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}] \\ x_i &\longmapsto \delta_i x_i + \sum_{F \in \mathcal{P}, i \in F} x_F \end{aligned}$$

makes $\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]$ an S_{M}- and S_{M}°-algebra. By the Feichtner–Yuzvinski presentation of $A^{\bullet}(\mathrm{M})$, we find

$$\mathrm{Tor}_0^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M}}], \mathbb{C})_{\bullet} \cong \frac{\mathbb{C}[\Sigma_{\mathrm{M}}]}{\mathrm{im}\,(S_{\mathrm{M}}^{\circ})} \cong A^{\bullet}(\mathrm{M}),$$

and so the higher Tor groups

$$\mathrm{Tor}_{>0}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M}}], \mathbb{C})_{\bullet}$$

measure the extra relations between the generators of $\mathrm{im}\,(S_{\mathrm{M}}^{\circ})$.

The second factor of all Tor groups will be $\mathbb{C}$, so we omit this factor from now on.

Example 2.8. Let us describe in detail $\Sigma_{\mathrm{M},\emptyset}$ and $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$.

The fan $\Sigma_{\mathrm{M},\emptyset} \subseteq \mathbb{R}^n/\mathbb{R} \cdot (1, \ldots, 1)$ has rays

$$\Sigma_{\mathrm{M},\emptyset}(1) = \{e_i : i \in [n]\},$$

and all cones of the fan are of the form $\sigma_{I<\emptyset}$ where I does not span $\widehat{1}$. In other words, $\mathrm{cone}(\{e_i : i \in I\}) \in \Sigma_{\mathrm{M},\emptyset}$ if and only if I does not contain a basis of M.

Therefore, the Stanley–Reisner ideal of $\Sigma_{\mathrm{M},\emptyset}$ is given by

$$I_{\Sigma_{\mathrm{M},\emptyset}} = \left\langle \prod_{i \in B} x_i : B \in \mathcal{B}(\mathrm{M}) \right\rangle.$$

This ideal appears in [9, 18] as the *facet ideal of the independence complex of* M.

2.3. **The Koszul Resolution.** The Koszul resolution of $\mathbb{C}$ gives an explicit way of computing the Tor algebra $\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}(-, \mathbb{C})_{\bullet}$. We define this resolution for any polynomial ring defined over $\mathbb{C}$.

Let $S = \mathbb{C}[x_1, \ldots, x_n]$, A an S-module, and $\mathbb{C}$ the residue field of the maximal homogeneous ideal of S. Let L be the ideal of S generated by degree 1 elements. Then

$$0 \to \bigwedge^{n-1} L \to \cdots \to \bigwedge^{2} L \to L \to 0 \tag{2}$$

is a free resolution of $\mathbb{C}$ in the category of S-modules. Explicitly, the differential applied to an element $\ell_1 \wedge \cdots \wedge \ell_k$ is defined as

$$d\left(\ell_1 \wedge \cdots \wedge \ell_k\right) = \sum_{j=1}^{k} (-1)^j \ell_j \cdot \left(\ell_1 \wedge \cdots \wedge \widehat{\ell_j} \wedge \cdots \ell_k\right).$$

Let $\mathcal{K}^\bullet(A)_S$ be the complex obtained by tensoring (over S) the Koszul resolution in (2) with A. We then identify the Koszul cohomology $H^t\left(\mathcal{K}^\bullet(A)_S\right)$ with the Tor groups $\operatorname{Tor}_t^S(A)$.

We now use these Koszul resolutions to prove a Künneth formula for the Tor algebra of tensor products of modules over polynomial rings. We use this heavily in Section 5.

Definition 2.9. Let S and R be polynomial rings over $\mathbb{C}$, and suppose $S \to A$ and $R \to B$ are S- and R-modules. Define the tensor module $A \otimes_{\mathbb{C}} B$ over $S \otimes_{\mathbb{C}} R$ from the map $S \times R \to A \times B$.

For complexes $(A^\bullet, d)$ and $(B^\bullet, d')$, define the tensor complex $((A \otimes B)^\bullet, \delta)$ by

$$(A \otimes B)^t = \bigoplus_{t_1+t_2=t} A^{t_1} \otimes_{\mathbb{C}} B^{t_2}$$

and differential

$$\begin{aligned} \delta\colon A^{t_1} \otimes B^{t_2} &\longrightarrow A^{t_1-1} \otimes B^{t_2} \oplus A^{t_1} \otimes B^{t_2-1} \\ \delta(a \otimes b) &\longmapsto da \otimes b + (-1)^{t_1} a \otimes d'b. \end{aligned}$$

Lemma 2.10. *Let S and R be polynomial rings over $\mathbb{C}$, and let A, B be S- and R-modules respectively. The map*

$$\begin{aligned} (\mathcal{K}(A)_S \otimes \mathcal{K}(B)_R)^\bullet &\longrightarrow \mathcal{K}^\bullet(A \otimes_{\mathbb{C}} B)_{S \otimes_{\mathbb{C}} R} \\ \left(a \otimes \bigwedge \ell_i\right) \otimes \left(b \otimes \bigwedge \mu_j\right) &\longmapsto (a \otimes b) \otimes \left(\bigwedge \ell_i \wedge \bigwedge \mu_j\right) \end{aligned}$$

is an isomorphism between the tensor complex of Koszul complexes and the Koszul complex of the tensor module. In particular,

$$\bigoplus_{t_1+t_2=t} H^{t_1}(\mathcal{K}^\bullet(A)_S) \otimes_{\mathbb{C}} H^{t_2}(\mathcal{K}^\bullet(B)_R) \cong H^t(\mathcal{K}^\bullet(A \otimes_{\mathbb{C}} B)_{S \otimes_{\mathbb{C}} R}).$$

Proof. Define an ordered basis of the degree 1 elements of $S \otimes_{\mathbb{C}} R$ by taking an ordered basis of the degree 1 elements of S followed by an ordered basis of the degree 1 elements of R. Then the isomorphism

$$(\mathcal{K}(A)_S \otimes \mathcal{K}(B)_R)^\bullet \longrightarrow \mathcal{K}^\bullet(A \otimes_{\mathbb{C}} B)_{S \otimes_{\mathbb{C}} R}$$

follows immediately from the definition of the Koszul complex. Alternatively, this statement is [8, Proposition 17.9]. The conclusion about the isomorphism of cohomology groups is the Künneth formula for complexes of vector spaces [25, Theorem 3.6.3]. □

2.4. **Hilbert Series.** Returning to the main case of the S_{M}- and S_{M}°- algebras $\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]$, the Koszul construction makes it clear that the Tor algebras have a natural bigrading

$$\mathrm{Tor}_t^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]\right)_s \text{ and } \mathrm{Tor}_t^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]\right)_s.$$

The grading t, or *Tor degree*, comes from indexing of the Tor functors, while the grading s, or *Stanley–Reisner degree*, is inherited from the grading of $\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]$ as a polynomial ring. Therefore the differential of the Koszul complex is homogenous of degree -1 in Tor degree and degree 1 in Stanley–Reisner degree.

The complex dimension of each bigraded piece is called a *Betti number* of $\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]$. Their generating function is the *Hilbert series*.

Definition 2.11. Let $V_{\bullet,\bullet} = \bigoplus_{i,j} V_{i,j}$ be a bigraded vector space. The *Hilbert series of* V is

$$\mathrm{Hilb}\left(V_{\bullet,\bullet}\right) = \sum_{i,j} \dim V_{i,j} x^i y^j.$$

So in our bigrading of the Tor algebra we have

$$\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]\right)_{\bullet}\right) = \sum_{i,j} \dim \mathrm{Tor}_i^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]\right)_j x^i y^j.$$

As a first computation, we calculate $\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M}}]\right)_{\bullet}\right)$ for rank 1 matroids.

Proposition 2.12. *Let* M *be a rank* 1 *matroid on* $[n]$*. Then*

$$\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M}}]\right)_{\bullet}\right) = (1+x)^{n-1}.$$

Proof. As an S_{M}°-algebra, $\mathbb{C}[\Sigma_{\mathrm{M}}] \cong \mathbb{C}$. The Koszul complex $\mathcal{K}^{\bullet}(\mathbb{C})_{S_{\mathrm{M}}^{\circ}}$ has vanishing differentials, and $\dim \mathcal{K}^i(\mathbb{C})_{S_{\mathrm{M}}^{\circ}} = \binom{n-1}{i}$, concentrated in Stanley–Reisner degree 0. ☐

In Section 4 we will need the fact that $\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet}\right)$ is a polynomial and not just a (potentially infinite) power series. While this is an immediate implication of Franz's result identifying $\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet}$ with the cohomology ring of $X(\Sigma_{\mathrm{M},\emptyset})$, we prefer to give a proof in terms of the Koszul complex because it is of combinatorial interest. The main idea is that just as computations involving the Stanley–Reisner ring may be reduced to computations involving only square-free monomials, the multilinear analogue is true for the Koszul complex.

To describe this, we must use an explicit basis of $\mathcal{K}^{\bullet}(\Sigma_{\mathrm{M},\emptyset})_{S_{\mathrm{M}}^{\circ}}$ over $\mathbb{C}$. For a vector $\ell \in \mathbb{Z}_{\geq 0}^n$, write $\ell = (\ell_1, \ldots, \ell_n)$, and define the monomial

$$x_\ell = \prod_{i=1}^{n} x_i^{\ell_i}.$$

We define the *support* of ℓ to be

$$\mathrm{supp}(\ell) = \{i \in [n] : \ell_i \neq 0\}.$$

Note that $x_\ell \neq 0$ in $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$ if and only if $\mathrm{supp}(\ell)$ contains no basis of M, or equivalently $\mathrm{rank}_{\mathrm{M}}(\mathrm{supp}(\ell)) < r$. The set $\left\{x_\ell : \ell \in \mathbb{Z}^n_{\geq 0} \text{ and } \mathrm{rank}_{\mathrm{M}}(\mathrm{supp}(\ell)) < r\right\}$ is a basis of $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$.

For each ℓ with $\mathrm{rank}_{\mathrm{M}}(\mathrm{supp}(\ell)) < r$, we construct an ordered basis for the degree 1 elements of S°_{M}. Let $\ell_+ = \max([n] \setminus \mathrm{supp}(\ell))$; this exists because $\mathrm{supp}(\ell) \neq [n]$. We then have the basis of L

$$\left\{x_i - x_{\ell_+} : i \in [n] \setminus \ell_+\right\},$$

and this induces the basis of $\bigwedge^\bullet L$

$$\mathcal{B}_\ell = \left\{(x_{i_1} - x_{\ell_+}) \wedge \cdots \wedge (x_{i_t} - x_{\ell_+}) : i_1 < \cdots < i_t\right\}.$$

A basis for $\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ is

$$\{x_\ell \otimes \xi : \mathrm{rank}_{\mathrm{M}}(\mathrm{supp}(\ell)) < r \text{ and } \xi \in \mathcal{B}_\ell\}.$$

We now come to the square-free analogue of the Koszul complex. A basis element $x_\ell \otimes \xi$ is *square-free* if $\max_{i\in[n]} \ell_i \leq 1$ and $\xi = (x_{i_1} - x_{\ell_+}) \wedge \cdots \wedge (x_{i_t} - x_{\ell_+})$ with each $i_k \notin \mathrm{supp}(\ell)$.

The span of the square-free basis elements forms a subspace we denote $\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$, and we want to show this is a subcomplex as well. It is straightforward that the differential of an element of $\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ can be written as the linear combination of elements $x_\ell \otimes \xi$ such that $\max_{i\in[n]} \ell_i \leq 1$. It remains to verify that no $x_i - x_{\ell_+}$ with $i \in \mathrm{supp}(\ell)$ occurs in ξ. For this we need the following lemma.

Lemma 2.13. *Suppose $z \in \mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ satisfies $dz \in \mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$. For $j \in [n]$, let k be the maximum number such that x_j^k occurs in a non-zero term of the basis representation of z; assume $k > 0$. Writing z in terms of the basis*

$$z = \sum_\ell \sum_{\xi \in \mathcal{B}_\ell} c_{\ell,\xi} \cdot x_\ell \otimes \xi,$$

if $\ell_j \geq k$ and $(x_j - x_{\ell_+})$ appears in ξ, then $c_{\ell,\xi} = 0$.

Proof. We consider the basis elements with the maximal number of x_j terms appearing:

$$z = \sum_{\{\ell : \ell_j = k\}} \sum_{\substack{\xi \in \mathcal{B}_\ell \\ \text{containing } (x_j - x_{\ell_+})}} c_{\ell,\xi} \cdot x_\ell \otimes \xi + \text{other terms}.$$

Taking the differential, we find

$$dz = \sum_{\{\ell : \ell_j = k\}} \sum_{\substack{\xi \in \mathcal{B}_\ell \\ \text{containing } (x_j - x_{\ell_+})}} \pm c_{\ell,\xi} \cdot x_{\ell+e_j} \otimes \xi \setminus (x_j - x_{\ell_+}) + \text{terms not involving } x_j^{k+1}.$$

Here $e_j \in \mathbb{Z}^n_{\geq 0}$ is 1 in the j-th coordinate and 0 elsewhere, and $\xi \setminus (x_j - x_{\ell_+})$ is the element of $\mathcal{B}_\ell$ obtained by removing the factor of $(x_j - x_{\ell_+})$ from ξ. The assumption that $k > 0$ implies $\mathrm{supp}(\ell + e_j) = \mathrm{supp}(\ell)$ whenever $\ell_j = k$. In particular,

$$\sum_{\{\ell : \ell_j = k\}} \sum_{\substack{\xi \in \mathcal{B}_\ell \\ \text{containing } (x_j - x_{\ell_+})}} \pm c_{\ell,\xi} \cdot x_{\ell+e_j} \otimes \xi \setminus (x_j - x_{\ell_+})$$

is a sum of non-square-free basis elements, so $c_{\ell,\xi} = 0$ if $\ell_j \geq k$ and $(x_j - x_{\ell_+})$ appears in ξ. ☐

Lemma 2.14. *The subspace* $\mathcal{K}^\bullet_{sf}\left(\Sigma_{\mathrm{M},\emptyset}\right)_{S^\circ_{\mathrm{M}}}$ *is a subcomplex.*

Proof. Let $z \in \mathcal{K}^\bullet_{sf}\left(\Sigma_{\mathrm{M},\emptyset}\right)_{S^\circ_{\mathrm{M}}}$ and dz its differential. Then dz satisfies the assumptions of the previous lemma because $ddz = 0$. Above we noted that dz can be written as the linear combination of elements $x_\ell \otimes \xi$ such that $\max_{i\in[n]} \ell_i \leq 1$ and that it remains to verify that no $x_i - x_{\ell_+}$ with $i \in \operatorname{supp}(\ell)$ occurs in ξ. Applying the lemma for all $i \in [n]$ verifies this condition. ☐

In contrast to $\mathcal{K}^\bullet\left(\Sigma_{\mathrm{M},\emptyset}\right)_{S^\circ_{\mathrm{M}}}$, the complex $\mathcal{K}^\bullet_{sf}\left(\Sigma_{\mathrm{M},\emptyset}\right)_{S^\circ_{\mathrm{M}}}$ is finite-dimensional; however, they are quasi-isomorphic.

Proposition 2.15. *The natural inclusion of complexes* $\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}} \hookrightarrow \mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ *induces an isomorphism*

$$H^\bullet(\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}) \cong H^\bullet(\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}).$$

In particular $\operatorname{Tor}^{S^\circ_{\mathrm{M}}}_\bullet\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet$ *is finite-dimensional.*

Proof. We will prove that any $z \in \mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ satisfying $dz \in \mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ is cohomologous to an element of $\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$. This statement is sufficient. Indeed, it implies any cycle of $\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ is cohomologous to a cycle of $\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$, and therefore implies the surjectivity of the map

$$H^\bullet(\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}) \twoheadrightarrow H^\bullet(\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}).$$

Moreover, if a cycle $z \in \mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ is a boundary in $\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$, then applying the statement to a lift of z in $\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ shows z is a boundary as well in $\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$. This implies the injectivity of the map

$$H^\bullet(\mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}) \hookrightarrow H^\bullet(\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}).$$

So suppose $z \in \mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ satisfies $dz \in \mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$, and write z in terms of the basis of $\mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$:

$$z = \sum_\ell \sum_{\xi \in \mathcal{B}_\ell} c_{\ell,\xi} \cdot x_\ell \otimes \xi.$$

This allows us to define the *excess degree* by

$$\operatorname{e.deg}(z) = \max\left\{(\max(\ell_1 - 1, 0), \ldots, \max(\ell_n - 1, 0)) : c_{\ell,\xi} \neq 0 \text{ for some } \xi\right\},$$

with the maximum taken with respect to the lexicographical ordering of $\mathbb{Z}^n_{\geq 0}$. The excess degree measures how far the non-zero x_ℓ appearing in z are from being square-free.

We claim that if $\operatorname{e.deg}(z) = \vec{0}$, then $z \in \mathcal{K}^\bullet_{sf}(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$. Indeed, $\operatorname{e.deg}(z) = \vec{0}$ implies that if $\max_{i\in[n]} \ell_i > 1$, then $c_{\ell,\xi} = 0$, and applying Lemma 2.13 for all $i \in [n]$ implies that if $i \in \operatorname{supp}(\ell)$ and $(x_i - x_{\ell_+})$ appears in ξ, then $c_{\ell,\xi} = 0$. Together, this says z is the sum of square-free basis elements.

Now assume $\operatorname{e.deg}(z) > \vec{0}$. By induction it suffices to show that z is cohomologous to $w \in \mathcal{K}^\bullet(\Sigma_{\mathrm{M},\emptyset})_{S^\circ_{\mathrm{M}}}$ with $\operatorname{e.deg}(w) < \operatorname{e.deg}(z)$.

Let j be the minimal element of $[n]$ such that $\mathrm{e.deg}(z)_j > 0$, and set $k = \mathrm{e.deg}(z)_j$. Define

$$v = \sum_{\{\ell : \ell_j = k\}} c_{\ell,\xi} \cdot x_{\ell - e_j} \otimes (x_j - \ell_+) \wedge \xi$$

We claim that $w = z - dv$ satisfies $\mathrm{e.deg}(w) < \mathrm{e.deg}(z)$. By construction, $\mathrm{e.deg}(w)_j < \mathrm{e.deg}(z)_j$, because dv cancels all terms in z containing x_j^k. For $i < j$, applying Lemma 2.13 to z shows dv contributes no square, or higher, powers of x_i. Thus $\mathrm{e.deg}(w)_i < \mathrm{e.deg}(z)_i$ as well. By the lexicographical ordering $\mathrm{e.deg}(w) < \mathrm{e.deg}(z)$, and this completes the proof. □

3. $\mathrm{Tor}_\bullet^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet$ via Hochster's Formula

In this section we determine

$$\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet\right)$$

by way of Hochster's formula. For this, it is traditional to work with simplicial complexes rather than fans.

Let Δ be a simplicial complex on $[n]$, and let $S_\Delta = \mathbb{C}[x_1, \ldots, x_n]$.

Definition 3.1. The *Stanley–Reisner ring of* Δ is

$$\mathbb{C}[\Delta] = \frac{S_\Delta}{\langle x_{i_1} \cdots x_{i_j} : \{i_1, \ldots, i_j\} \notin \Delta \rangle}.$$

Hochster's Formula describes how to compute the Betti numbers of the Stanley–Reisner ring of a simplicial complex in terms of its reduced cohomology. We recall the formula for the $\mathbb{Z}^2$-graded Betti numbers; many sources have a finer $\mathbb{Z} \times \mathbb{Z}^n$-grading for the Betti numbers based on the support of a representative in the cohomology of the Koszul complex, e.g. [21, Theorem 5.12]. In passing from the ring S_{M} to S_{M}° as we do in Section 4, this grading is lost, so we prefer not to introduce it.

Theorem 3.2 (Hochster's Formula)**.** *Let* Δ *be a simplicial complex on* $[n]$*. Then*

$$\dim \mathrm{Tor}_t^{S_\Delta}(\mathbb{C}[\Delta])_s = \sum_{\substack{W \subseteq [n] \\ |W| = t+s}} \dim \widetilde{H}^{s-1}(\Delta|_W; \mathbb{C}),$$

where $\Delta|_W$ *is the restriction of the simplicial fan* Δ *to the vertices in* W*.*

For this to be useful in our context, we start by turning $\Sigma_{\mathrm{M},\emptyset}$ into a simplicial complex.

Definition 3.3. Intersecting $\Sigma_{\mathrm{M},\emptyset}$ with the unit sphere yields the *non-spanning complex* $\mathrm{NS}(\mathrm{M})$. Concretely, $\mathrm{NS}(\mathrm{M})$ is a simplicial complex on $[n]$ whose faces are the *non-spanning* sets in $[n]$, that is, those subsets which do not span M. Note that as S_{M}-algebras $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}] \cong \mathbb{C}[\mathrm{NS}(\mathrm{M})]$.

Corollary 3.4. *Let* M *be a matroid. Then*

$$\dim \mathrm{Tor}_t^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s = \sum_{\substack{W \subseteq [n] \\ |W| = t+s}} \dim \widetilde{H}^{s-1}(\mathbb{C}[\mathrm{NS}(M)|_W]; \mathbb{C}).$$

We first compute the reduced cohomology for non-spanning complexes.

Proposition 3.5. *For matroid* M *of rank* r*,*

$$\dim \widetilde{H}^i\left(\mathrm{NS(M)}; \mathbb{C}\right) = \begin{cases} |\mathrm{nbc}\left(\mathrm{M}\right)| & i = r-2 \\ 0 & i \neq r-2. \end{cases}$$

Proof. The Alexander dual of NS(M) is the *independence complex* $\mathrm{IN}\left(\mathrm{M}^*\right)$ of the dual matroid. This is the complex on $[n]$ whose faces are independent sets of M^*; see [18, Proposition 1] for a proof. The reduced homology groups of this complex were computed in [2, Theorem 7.8.1] to be

$$\widetilde{H}_i\left(\mathrm{IN(M^*)}; \mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}^{|\mathrm{nbc(M)}|} & i = r-1 \\ 0 & i \neq r-1. \end{cases}$$

The result then follows from Combinatorial Alexander Duality [3, Theorem 1.1]

$$\widetilde{H}_i\left(\mathrm{NS(M)}; \mathbb{Z}\right) \cong \widetilde{H}^{n-i-3}\left(\mathrm{IN}\left(\mathrm{M}^*\right); \mathbb{Z}\right).$$

□

This allows us to compute each summand on the right-hand side of Corollary 3.4.

Proposition 3.6. *Let* M *be a matroid of rank* r *on* $[n]$ *and* $W \subseteq [n]$*.*

(1) *If* $W = \emptyset$*,*

$$\dim \widetilde{H}^{s-1}\left(\mathrm{NS(M)}|_W; \mathbb{C}\right) = \begin{cases} 0 & s \neq 0 \\ 1 & s = 0. \end{cases}$$

(2) *If* W *is non-empty and* $\mathrm{rank}_{\mathrm{M}}(W) < r$*, for all* s

$$\widetilde{H}^{s-1}\left(\mathrm{NS(M)}|_W; \mathbb{C}\right) = 0.$$

(3) *If* W *is non-empty and* $\mathrm{rank}_{\mathrm{M}}(W) = r$*,*

$$\dim \widetilde{H}^{s-1}\left(\mathrm{NS(M)}|_W; \mathbb{C}\right) = \begin{cases} 0 & s \neq r-1 \\ \left|\mathrm{nbc}\left(\mathrm{M}\,|_W\right)\right| & s = r-1. \end{cases}$$

Proof. (1) This is the computation of the reduced homology of the empty simplicial complex.

(2) In this case, $\mathrm{NS(M)}|_W$ is a non-empty simplex, and the reduced cohomology vanishes.

(3) If W has full rank, then $\mathrm{NS(M)}|_W$ is $\mathrm{NS}\left(\mathrm{M}\,|_W\right)$. The cohomology of this complex is computed in Corollary 3.5.

□

From Hochster's Formula and Proposition 3.6, we have the following Hilbert series.

Proposition 3.7. *For a matroid* M *of rank* r*,*

$$\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet}\right) = 1 + \sum_{\{W \subseteq [n] : \mathrm{rank}_{\mathrm{M}}(W) = r\}} \left|\mathrm{nbc}\left(\mathrm{M}\,|_W\right)\right| x^{|W|-r+1} y^{r-1}.$$

An application of Lemma 2.3 re-indexes the summation over bases of M.

Proposition 3.8. *For a matroid* M *of rank* r,

$$\operatorname{Hilb}\left(\operatorname{Tor}_\bullet^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet\right) = 1 + xy^{r-1} \sum_{B\in\mathcal{B}(\mathrm{M})} (1+x)^{\mathrm{ep}(B)}.$$

As $n - r - \mathrm{ea}(B) = \mathrm{ep}(B)$, we recognize this as a specialization of the Tutte polynomial.

Theorem 3.9. *For a matroid* M *of rank* r,

$$\operatorname{Hilb}\left(\operatorname{Tor}_\bullet^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet\right) - 1 = x\,(1+x)^{n-r}\, y^{r-1} T_{\mathrm{M}}\left(1, \frac{1}{1+x}\right).$$

Remark 3.10. Similar specializations of the Tutte polynomial have appeared in the coding theory literature, as we now summarize. For details and definitions in coding theory, see [5, Section 6.5].

Let C be a $[n,r]$-*linear code over* q and $\mathrm{M}(C)$ the associated matroid. The generating function for the number of codewords in C with given weight w is the *codeweight enumerator*

$$A_{C,q}(z) = \sum_{c\in C} z^{w(c)}.$$

Greene [16] showed that the codeweight enumerator is a specialization of the Tutte polynomial of $\mathrm{M}(C)$ (see [5, Proposition 6.5.1] for the statement we give),

$$A_{C,q}(z) = (1-z)^r z^{n-r} T_{\mathrm{M}(C)}\left(\frac{1+(q-1)z}{1-z}, \frac{1}{z}\right). \tag{3}$$

Another application of Greene's result implies there is a bivariate *extended weight enumerator polynomial* $W_C(y,z)$ with the property that

$$W_C(q^i, z) = A_{C_i, q^i}(z)$$

for $i \geq 1$. Here C_i is the extension of C over q^i. Theorem 3.9 says that for an $[n,r]$-linear code C (over any finite field),

$$\operatorname{Hilb}\left(\operatorname{Tor}_\bullet^{S_{\mathrm{M}(C)}}\left(\mathbb{C}[\Sigma_{\mathrm{M}(C),\emptyset}]\right)_\bullet\right)\Big|_{x=y=z-1} - 1 = (-1)^r W_C(0,z).$$

Johnsen, Roksvold, and Verdure obtain a similar formula for $W_C(0,z)$ in terms of the Betti numbers of the Stanley–Reisner ideal of $\mathrm{IN}\left(\mathrm{M}(C)^*\right)$ [19, Corollary 5.1].

4. $\operatorname{Tor}_\bullet^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet$ Via Flat Base Change

We now use flat base change and a long exact sequence to compute the Betti numbers of $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$ over S_{M}° from the Betti numbers over S_{M}.

As $S_{\mathrm{M}} \cong S_{\mathrm{M}}^\circ[x_n]$, the inclusion $S_{\mathrm{M}}^\circ \hookrightarrow S_{\mathrm{M}}$ is flat, so the following result on flat base change applies.

Proposition 4.1 (cf. [25, Proposition 3.2.9])**.** *If* $R \to S$ *is a flat ring map,* A *is an* R*-module, and* B *is an* S*-module, there is a natural isomorphism*

$$\operatorname{Tor}_\bullet^R(A,B) \cong \operatorname{Tor}_\bullet^S(A\otimes_R S, B)$$

For an auxilliary indeterminant y, we endow $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y]$ with the structure of an S_{M}-algebra via the map

$$\begin{aligned} S_{\mathrm{M}} &\longrightarrow \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y] \\ x_i &\longmapsto x_i + y. \end{aligned}$$

Lemma 4.2. *There is a degree-preserving isomorphism*

$$\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet} \cong \mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y]\right)_{\bullet}.$$

Proof. Apply flat base change to $R = S_{\mathrm{M}}^{\circ}$, $S = S_{\mathrm{M}}$, $A = \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$, and $B = \mathbb{C}$.

As S_{M}-algebras,

$$\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}] \otimes_{S_{\mathrm{M}}^{\circ}} S_{\mathrm{M}} \cong \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y]$$

via the S_{M}°-balanced map

$$\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}] \times S_{\mathrm{M}} \to \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y]$$

which is the natural inclusion on the first factor and the structure map on the second factor.

All of the isomorphisms preserve degree, so the result is proved. □

We now compare the S_{M}-algebras $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$ and $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y]$ through the short exact sequence

$$0 \to \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y] \xrightarrow{\cdot y} \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}][y] \to \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}] \to 0.$$

By the identification in Lemma 4.2, this produces the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Tor}_{\bullet+1}^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet} \to \mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet} \to \mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet+1} \\ \to \mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet+1} \to \mathrm{Tor}_{\bullet-1}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet+1} \to \cdots. \end{aligned} \tag{4}$$

In order to compute the kernel and cokernel of the maps

$$\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet} \to \mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet+1}$$

we use following vanishing lemma.

Proposition 4.3. *Let* M *be a matroid of rank* r*.*

(1) *For* $s > r-1$, $\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s = 0$.
(2) *For* $t > 0$ *and* $s < r-1$, $\mathrm{Tor}_{t}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s = 0$.

In particular, $\mathrm{Tor}_{>0}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{\bullet} = \mathrm{Tor}_{>0}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{r-1}$.

Proof. (1) For $t \geq 0$ such that $\mathrm{Tor}_{t}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{M,\emptyset}]\right)_{\bullet} \neq 0$, let s_t be the maximal integer with $\mathrm{Tor}_{t}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{s_t} \neq 0$ (the existence of such an integer is guaranteed by Proposition 2.15). Then the long exact sequence (4) implies the connecting homomorphism

$$\mathrm{Tor}_{t+1}^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{s_t} \twoheadrightarrow \mathrm{Tor}_{t}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{s_t}$$

is surjective, as $\mathrm{Tor}_{t}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{s_t+1} = 0$. As $\mathrm{Tor}_{t+1}^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{s} = 0$ for $s > r-1$, we must have $s_t \leq r-1$.

(2) For $t > 0$ such that $\operatorname{Tor}_t^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet \neq 0$, let s_t be the minimal integer with $\operatorname{Tor}_t^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{s_t} \neq 0$. The long exact sequence (4) implies the injectivity of the map

$$\operatorname{Tor}_t^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{s_t} \hookrightarrow \operatorname{Tor}_t^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{s_t}.$$

As $\operatorname{Tor}_t^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s = 0$ for $t > 0$ and $s < r-1$, we must have $s_t \geq r-1$. □

We now deduce the Betti numbers of $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$ over S_{M}°.

Theorem 4.4. *For a matroid* M *of rank* r,

$$\operatorname{Hilb}\left(\operatorname{Tor}_\bullet^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet\right) = \sum_{i=0}^{r-1} y^i + xy^{r-1} \sum_{B \in \mathcal{B}(\mathrm{M}) \setminus B_{\max}} (1+x)^{\mathrm{ep}(B)-1}.$$

Proof. By Proposition 4.3, the maps

$$\operatorname{Tor}_t^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet \to \operatorname{Tor}_t^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{\bullet+1}$$

in the long exact sequence (4) vanish for $t > 0$, because these Tor groups are supported only in Stanley–Reisner degree $r-1$. We therefore have the exact sequence

$$\begin{aligned}0 \to \operatorname{Tor}_1^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s \to \operatorname{Tor}_1^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s \to \operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s \\ \to \operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{s+1} \to \operatorname{Tor}_0^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{s+1} \to 0.\end{aligned} \tag{5}$$

By Proposition 3.8,

$$\operatorname{Tor}_1^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet = \operatorname{Tor}_1^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{r-1},$$

and

$$\operatorname{Tor}_0^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet = \operatorname{Tor}_0^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_0.$$

For $0 \leq s < r-1$, the exact sequence (5) reads

$$0 \to \operatorname{Tor}_1^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s \to 0 \to \operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s \to \operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{s+1} \to 0,$$

and we have an isomorphism

$$\operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s \cong \operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{s+1}.$$

For $s = -1$, the exact sequence (5) reads

$$0 \to 0 \to 0 \to 0 \to \operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_0 \to \operatorname{Tor}_0^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_0 \to 0,$$

and this gives an isomorphism

$$\operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_0 \cong \operatorname{Tor}_0^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_0 \cong \mathbb{C}.$$

By Proposition 4.3, $\operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s = 0$ for $s > r-1$, so we obtain

$$\dim \operatorname{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s = \begin{cases} 1 & 0 \leq s \leq r-1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying this to the first part of the exact sequence (5),

$$\dim \mathrm{Tor}_1^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s \oplus \mathrm{Tor}_0^{S_{\mathrm{M}}^\circ}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s = \begin{cases} \dim \mathrm{Tor}_1^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s + 1 & 0 \le s < r-1 \\ \dim \mathrm{Tor}_1^{S_{\mathrm{M}}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_s & \text{otherwise.} \end{cases}$$

From the second part of (5),

$$\dim \mathrm{Tor}_0^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s + \mathrm{Tor}_{-1}^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s = \begin{cases} \dim \mathrm{Tor}_0^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s + 1 & 0 < s \le r-1 \\ \dim \mathrm{Tor}_0^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s & \text{otherwise.} \end{cases}$$

Similarly for $t > 1$, we have the short exact sequence

$$0 \to \mathrm{Tor}_t^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet \to \mathrm{Tor}_t^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet \to \mathrm{Tor}_{t-1}^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet \to 0,$$

and this implies

$$\dim \mathrm{Tor}_t^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s + \mathrm{Tor}_{t-1}^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s = \dim \mathrm{Tor}_t^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s$$

for $t > 1$.

Altogether, these equations yield the relation of Hilbert series

$$\begin{aligned}
&(1+x)\,\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet\right) \\
&\qquad = \sum_{i=1}^{r-1} y^i + x\sum_{i=0}^{r-2} y^i + \mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_\bullet\right) \\
&\qquad = \sum_{i=1}^{r-1} y^i + x\sum_{i=0}^{r-2} y^i + 1 + xy^{r-1}\sum_{B\in\mathcal{B}(\mathrm{M})} (1+x)^{\mathrm{ep}(B)} \\
\text{(Lemma 2.4)}\quad &\qquad = \sum_{i=1}^{r-1} y^i + x\sum_{i=0}^{r-2} y^i + 1 + xy^{r-1}\sum_{B\in\mathcal{B}(M)\setminus B_{\max}} (1+x)^{\mathrm{ep}(B)} + xy^{r-1} \\
&\qquad = (1+x)\sum_{i=0}^{r-1} y^i + xy^{r-1}\sum_{B\in\mathcal{B}(M)\setminus B_{\max}} (1+x)^{\mathrm{ep}(B)}
\end{aligned}$$

Now divide both sides by $1+x$. □

In the case M is a loopless matroid of rank 2, there is an isomorphism of S_{M}°-algebras $\mathbb{C}[\Sigma_{\mathrm{M}}] \cong \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$. This leads to the full description of the Tor algebra of such matroids.

Corollary 4.5. *For a loopless matroid* M *of rank* 2*,*

$$\mathrm{Hilb}\left(\mathrm{Tor}^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M}}]\right)\right) = 1 + y + xy\sum_{B\in\mathcal{B}(\mathrm{M})\setminus B_{\max}} (1+x)^{\mathrm{ep}(B)-1}.$$

For loopless matroids of arbitrary rank, Theorem 4.4 yields a combinatorial count for $\dim \mathrm{Tor}_{n-r}^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{r-1}$.

Corollary 4.6. *Let* M *be a loopless matroid of rank* r *on ground set* $[n]$*. For* $t > n-r$ *or* $s > r-1$*,* $\mathrm{Tor}_t^{S_{\mathrm{M}}^\circ}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_s = 0$*.*

In top degree,

$$\dim \operatorname{Tor}_{n-r}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{r-1} = |\mathrm{nbc}(\mathrm{M})| \neq 0.$$

Even though we have an explicit count of dimension, it is not clear how use the previous proof to construct nice generators for $\operatorname{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{\bullet}$, realized as the cohomology of the Koszul complex $H^{\bullet}(\mathcal{K}^{\bullet}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{S_{\mathrm{M}}^{\circ}})$. One must find explicit bases for $\widetilde{H}_{s-1}(\mathrm{NS}(\mathrm{M})|_{W};\mathbb{C})$ and then run these through the isomorphism of Hochster's Formula and the exact sequences comparing the Tor groups over S_{M} and S_{M}°. However, one can explicitly construct a monomial basis in the case of Tor_1. This computation will be important in the next section when discussing the boundary map of the long exact sequence of matroidal flips.

Lemma 4.7. *Let* M *be a loopless matroid of rank* r. *The set*

$$\left\{ \prod_{k\in B\setminus i_B} x_k \otimes (x_{i_B} - x_{j_B}) : B \in \mathcal{B}(\mathrm{M}) \setminus B_{\max},\ j_B = \max \mathrm{EP}(B),\ i_B = \min\left(C_{B\cup j_B}\right) \right\}.$$

is a basis for $H^1(\mathcal{K}^{\bullet}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{S_{\mathrm{M}}^{\circ}}) \cong \operatorname{Tor}_1^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{\bullet}$.

Proof. Let A denote the proposed basis, and write an element of A as

$$\eta_B = \prod_{k\in B\setminus i_B} x_k \otimes (x_{i_B} - x_{j_B}).$$

Note that by Lemma 2.4, each η_B is non-zero (where $B \neq B_{\max}$). The set A has cardinality $|\mathcal{B}(\mathrm{M})| - 1$, so Theorem 4.4 implies it has the cardinality of a basis. It therefore suffices to show the elements of A are linearly independent. We will do this by finding a linear map of rank $|\mathcal{B}(\mathrm{M})| - 1$ on A.

Consider the short exact sequence defining the Stanley–Reisner ring,

$$0 \to I_{\Sigma_{\mathrm{M},\emptyset}} \to S_{\mathrm{M}} \to \mathbb{C}[\Sigma_{\mathrm{M},\emptyset}] \to 0.$$

It is an exact sequence of S_{M}°-algebas. Because $I_{\Sigma_{\mathrm{M},\emptyset}}$ is supported only in Stanley–Reisner degree r or higher, and the number of distinct degree r monomials in $I_{\Sigma_{\mathrm{M},\emptyset}}$ is $|\mathcal{B}(\mathrm{M})|$, it follows immediately from the degree of the differential in $\mathcal{K}^{\bullet}(I_{\Sigma_{\mathrm{M},\emptyset}})_{S_{\mathrm{M}}^{\circ}}$ that

$$\dim \operatorname{Tor}_0^{S_{\mathrm{M}}^{\circ}}\left(I_{\Sigma_{\mathrm{M},\emptyset}}\right)_r = |\mathcal{B}(\mathrm{M})|.$$

In terms of Koszul cohomology, $H^0(\mathcal{K}^{\bullet}(I_{\Sigma_{\mathrm{M},\emptyset}})_{S_{\mathrm{M}}^{\circ}})$ has a basis

$$\left\{ x_B := \prod_{i\in B} x_i : B \in \mathcal{B}(\mathrm{M}) \right\}.$$

We claim the connecting homomorphism

$$\delta\colon H^1\left(\mathcal{K}^{\bullet}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{S_{\mathrm{M}}^{\circ}}\right) \to H^0\left(\mathcal{K}^{\bullet}(I_{\Sigma_{\mathrm{M},\emptyset}})_{S_{\mathrm{M}}^{\circ}}\right)$$

has rank $|\mathcal{B}(\mathrm{M})| - 1$ on A.

From the description of the connecting homomorphism via the snake lemma,

$$\delta(\eta_B) = x_B - x_{B\cup j_B\setminus i_B}.$$

The external passivity of j_B implies that $B < B \cup j_B \setminus i_B$ in lexicographic order. For any non-trivial linear combination $\sum_{B \in \mathcal{B} \setminus B_{\max}} \alpha_B \eta_B$, take B' to be the minimal basis such that $\alpha_{B'} \neq 0$. Then

$$\delta \left(\sum_{B \in \mathcal{B} \setminus B_{\max}} \alpha_B \eta_B \right) = \alpha_{B'} x_{B'} + \text{terms involving larger bases} \neq 0.$$

Therefore δ has rank $|\mathcal{B}(\mathrm{M})| - 1$ on A, and this proves the result. $\square$

5. Matroidal Flips and the Tor Algebra of $\mathbb{C}[\Sigma_{\mathrm{M}}]$

Having computed $\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet}\right)$, we now turn towards our goal of understanding the Tor algebra $\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]\right)_{\bullet}$. We do this by gradually modifying the fan $\Sigma_{\mathrm{M},\emptyset}$ to become the fan $\Sigma_{\mathrm{M},\emptyset}$. As a result, the Stanley–Reisner ring $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$ is modified to become the ring $\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}]$. This section relies heavily on the theory of matroidal flips introduced in [1].

Definition 5.1. For two order filters (recall Definition 2.5) differing by a flat, say $Z = \mathcal{P}_+ \setminus \mathcal{P}_-$, the tropical modification $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ is called a *matroidal flip*. The flat Z is called the *center* of the matroidal flip.

If $\mathcal{P}_-$ is an order filter, we can construct a matroidal flip by taking Z to be any maximal element of $\widehat{\mathcal{L}}(\mathrm{M}) \setminus \mathcal{P}_-$ and then defining $\mathcal{P}_+ = \mathcal{P}_- \cup Z$. In particular, there is a series of matroidal flips interpolating between $\Sigma_{\mathrm{M},\emptyset}$ and Σ_{M}.

It is known that Chow rings decompose well under matroidal flips, and the goal of this section is to extend this to the full Tor algebra. Following the notation of [1], define

$$A^{\bullet}(\mathrm{M}, \mathcal{P}) := \mathrm{Tor}_0^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}}]\right)_{\bullet}$$

and

$$A^{\bullet}(\mathrm{M}) := \mathrm{Tor}_0^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M}}]\right)_{\bullet}.$$

The authors of [1] use these matroidal flips to construct a short exact sequence of Chow rings

$$0 \to A^{\bullet}(\mathrm{M}, \mathcal{P}_-) \to A^{\bullet}(\mathrm{M}, \mathcal{P}_+) \to \mathrm{E}_{0,\bullet}^{Z} \to 0,$$

where

$$\mathrm{E}_{0,s}^{Z} = \bigoplus_{\substack{s_1 + s_2 = s \\ s_1 > 0}} A^{s_1}(\mathrm{M}^Z, \emptyset) \otimes A^{s_2}(\mathrm{M}_Z).$$

An induction on the ground set of M and on order filters allows one to deduce properties of $A^{\bullet}(\mathrm{M})$, such as its Hilbert series, from those of $A^{\bullet}(\mathrm{M}, \emptyset)$.

We show this short exact sequence extends to a (not necessarily split) long exact sequence of Tor groups

$$\cdots \to \mathrm{E}_{\bullet+1,\bullet-1}^{Z} \to \mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_{\bullet} \to \mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_{\bullet} \to \mathrm{E}_{\bullet,\bullet}^{Z} \to \cdots. \tag{6}$$

Here

$$\mathrm{E}^Z_{t,s} = \bigoplus_{\substack{t_1+t_2=t\\ s_1+s_2=s\\ t_1+s_1>0}} \mathrm{Tor}^{S^\circ_{\mathrm{M}^Z}}_{t_1}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \mathrm{Tor}^{S^\circ_{\mathrm{M}_Z}}_{t_2}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_2}.$$

The proof of this fact is motivated by topology, and this reinforces the perspective of viewing Tor algebras as cohomology rings. We study how the Tor algebra changes under a matroidal flip by understanding how the topology of $X(\Sigma_{\mathrm{M},\mathcal{P}_-})$ compares to $X(\Sigma_{\mathrm{M},\mathcal{P}_+})$. Matroidal flips behave like (or in some cases exactly are) blow ups, so we can use the standard technique of computing the cohomology of a blow up by comparing appropriate Mayer–Vietoris sequences of the cohomology of the blow up and of the base space (see e.g. [17, pgs. 473-474]) to produce the long exact sequence (6). In this description, $\mathrm{E}^Z_{\bullet,\bullet}$ plays the role of the cohomology ring of the exceptional divisor modulo the cohomology ring of the center of the blow up (*i.e.*, the subvariety we blow up along).

5.1. **The Cohomology of Blow Ups.** Before discussing the case of matroidal flips, let us consider the topological picture of a blow up and recall the comparison of Mayer–Vietoris sequences expressing the cohomology of the blow up in terms of the cohomology of the base space, center, and exceptional divisor.

Let X be a nice topological space, X' the blow up along the closed center $Z \subseteq X$, and E the exceptional divisor. Let U_- be a small neighborhood of Z, and U_+ the preimage of U_- in X'; it is a tubular neighborhood of E. See Figure 1 for an example.

The coverings $\{X' \setminus E, U_+\}$ of X' and $\{X \setminus Z, U_-\}$ of X induce the Mayer–Vietoris exact sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{\bullet+1}(U_+ \setminus E) & \longrightarrow & H^\bullet(X') & \longrightarrow & H^\bullet(X'\setminus E) \oplus H^\bullet(U_+) & \longrightarrow & H^\bullet(U_+\setminus E) & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \cdots \longrightarrow & H^{\bullet+1}(U_- \setminus Z) & \longrightarrow & H^\bullet(X) & \longrightarrow & H^\bullet(X\setminus Z) \oplus H^\bullet(U_-) & \longrightarrow & H^\bullet(U_-\setminus Z) & \longrightarrow \cdots. \end{array}$$

There are homeomorphisms $X'\setminus E \cong X \setminus Z$ and $U_+\setminus E \cong U_-\setminus Z$ and retracts of U_+ onto E and U_- onto Z. Moreover, we have an injection $H^\bullet(Z) \hookrightarrow H^\bullet(E)$, as the cohomology of the exceptional divisor is the cohomology of projective space tensor the cohomology of the center. Thus, the differences between the exact sequences only involve the cohomology of X, X', Z, and E. The following lemma from homological algebra demonstrates how to compare these cohomology rings using a long exact sequence.

Lemma 5.2. *Suppose $A^\bullet_-$, $B^\bullet_-$, and $C^\bullet_-$ and $A^\bullet_+$, $B^\bullet_+$, and $C^\bullet_+$ are cocomplexes such that $\epsilon\colon B^\bullet_- \to B^\bullet_+$ is an injection, $\phi\colon C^\bullet_- \to C^\bullet_+$ is an isomorphism, and there is a commutative diagram of long exact sequences*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & C^{\bullet+1}_+ & \xrightarrow{\gamma_+} & A^\bullet_+ & \xrightarrow{\alpha_+} & B^\bullet_+ & \xrightarrow{\beta_+} & C^\bullet_+ & \xrightarrow{\gamma_+} \cdots \\ & \phi\uparrow & & \iota\uparrow & & \epsilon\uparrow & & \phi\uparrow & \\ \cdots \longrightarrow & C^{\bullet+1}_- & \xrightarrow{\gamma_-} & A^\bullet_- & \xrightarrow{\alpha_-} & B^\bullet_- & \xrightarrow{\beta_-} & C^\bullet_- & \xrightarrow{\gamma_-} \cdots. \end{array}$$

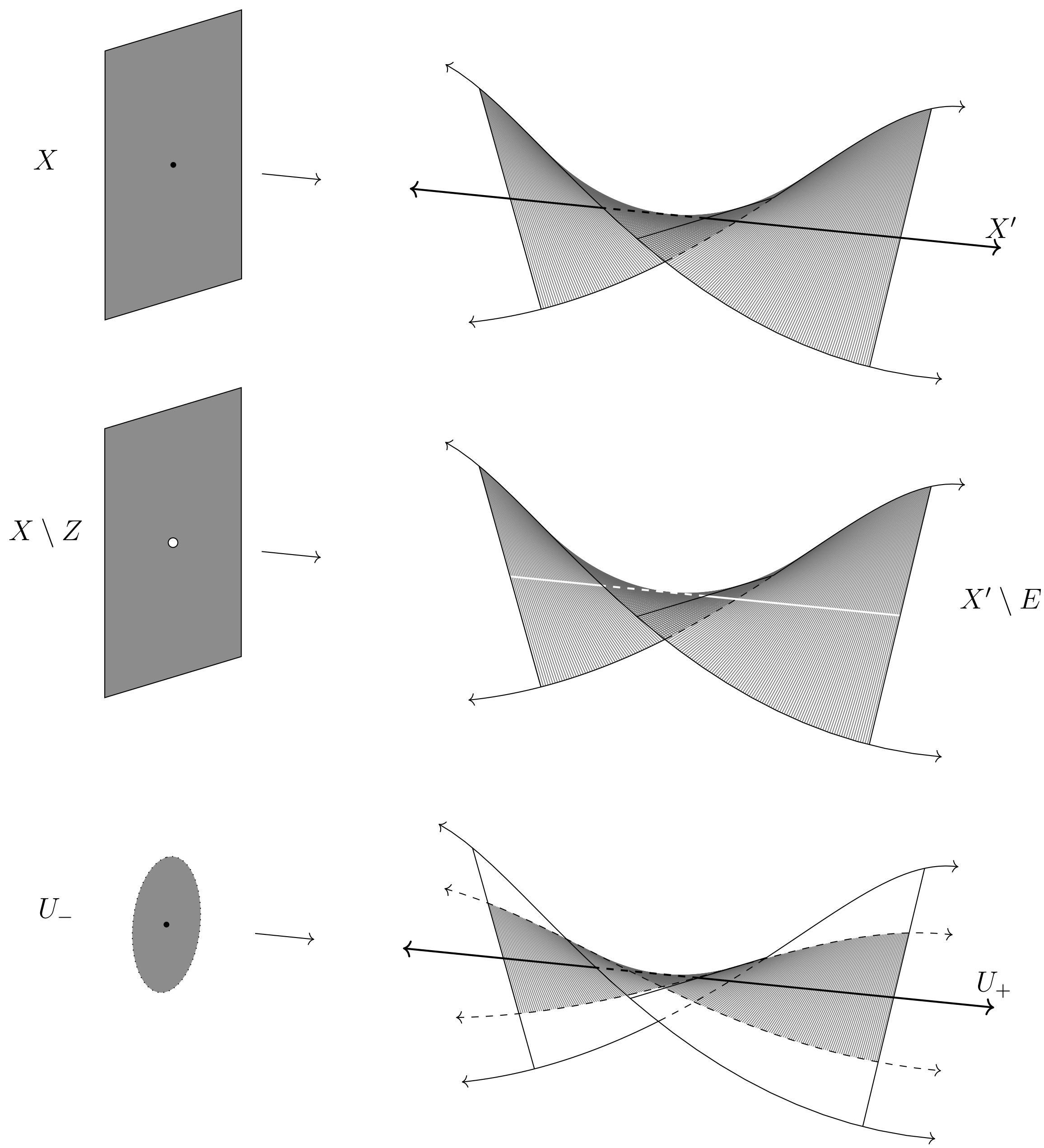

Figure 1. A Blow Up along a Point and the Open Covering

Then there is a long exact sequence

$$\cdots \to \operatorname{coker}(B_-^{\bullet+1} \to B_+^{\bullet+1}) \xrightarrow{\gamma_- \circ \phi^{-1} \circ \beta_+} A_-^\bullet \xrightarrow{\iota} A_+^\bullet \xrightarrow{\alpha_+} \operatorname{coker}(B_-^\bullet \to B_+^\bullet) \to \cdots .$$

Proof. Note this is well-defined as $\gamma_- \circ \beta_- \colon B_-^\bullet \to A_-^{\bullet-1}$ is the zero map. The proof of exactness is ultimately a diagram chase, and we include it for completeness.

Exactness at $A_-^\bullet$: From the commutativity of the diagram, the map

$$\iota \circ \gamma_- \circ \phi_{-1} \circ \beta_+ \colon \operatorname{coker}(B_-^{\bullet+1} \to B_+^{\bullet+1}) \to A_+^\bullet$$

is the map $\gamma_+ \circ \beta_+$, which is zero by the exactness of the top row. Now suppose $a \in \ker \iota$. From commutativity of the diagram and injectivity of ϵ, $a \in \ker \alpha_-$. The exactness of the bottom row allows us to find $c \in C_-^{\bullet+1}$ such that $\gamma_-(c) = a$. Then $\gamma_+ \circ \phi(c) = 0$, so the exactness of the top row allows us to find $b \in B_+^{\bullet+1}$ such that $\beta_+(b) = \phi(c)$. Therefore $\gamma_- \circ \phi^{-1} \circ \beta_+(b) = a$.

Exactness at $A_+^\bullet$: Commutativity implies that $\alpha_+ \circ \iota \colon A_-^\bullet \to \operatorname{coker}(B_-^\bullet \to B_+^\bullet)$ factors through $\epsilon \colon B_-^\bullet \to B_+^\bullet$ and is therefore the zero map. Now take $a_+ \in \ker(\alpha_+ \colon A_+^\bullet \to \operatorname{coker}(B_-^\bullet \to B_+^\bullet))$. There is some $b \in B_-^\bullet$ with $\epsilon(b) = \alpha_+(a)$. The injectivity of ϕ and commutativity of the diagram implies $\beta_-(b) = 0$, so we may find $a_- \in A_-^\bullet$ such that $\alpha_-(a_-) = b$. It may not be the case that $\iota(a_-) = a_+$, but $\alpha_+(a_+ - \iota(a_-)) = 0$, so there is some $c \in C_+^{\bullet+1}$ with $\gamma_+(c) + \iota(a_-) = a_+$. Then $\gamma_- \circ \phi^{-1}(c) + a_- \in A_-^\bullet$, and $\iota(\gamma_- \circ \phi^{-1}(c) + a_-) = a_+$.

Exactness at $\operatorname{coker}(B_-^\bullet \to B_+^\bullet)$: Exactness of the top row implies $\beta_+ \circ \alpha_+ = 0$, so

$$\gamma_- \circ \phi^{-1} \circ \beta_+ \circ \alpha_+ \colon A_+^\bullet \to A_-^{\bullet-1}$$

is the zero map. Now suppose $b_+ \in \ker(\gamma_- \circ \phi^{-1} \circ \beta_+)$. Then $\phi^{-1} \circ \beta_+(b_+) \in \ker(\gamma_-)$, so we may find $b_- \in B_-^\bullet$ with $\beta_-(b_-) = \phi^{-1} \circ \beta_+(b_+)$. Then $b_+ - \epsilon(b_-) \in \ker(\beta_+)$, so we find $a \in A_+^\bullet$ such that $\alpha_+(a) = b_+ - \epsilon(b_-)$. Finally, note that b_+ and $b_+ - \epsilon(b_-)$ represent the same element in $\operatorname{coker}(B_-^\bullet \to B_+^\bullet)$. □

This produces the long exact sequence for the cohomology of a blow up

$$\cdots \to H^\bullet(X) \to H^\bullet(X') \to \operatorname{coker}(H^\bullet(Z) \to H^\bullet(E)) \to \cdots.$$

In the case of a blow up, the map $H^\bullet(X) \to H^\bullet(X')$ is also injective, so the long exact sequence splits, and the cohomology of the blow up is relatively easy to compute. This will be in distinction with the long exact sequence of a general matroidal flip which does not split.

5.2. **Matroidal Flips as Blow Ups.** We will now see how matroidal flips are close enough to blow ups that we may apply an analogue of the previous argument. The most illustrative case is when the center $Z = \mathcal{P}_+ \setminus \mathcal{P}_-$ of the matroidal flip has M^Z a boolean matroid.

Example 5.3. Suppose the matroidal flip $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ has center Z such that M^Z is a boolean matroid.

Recall the description of Bergman fans $\Sigma_{\mathrm{M},\mathcal{P}}$ given in Definition 2.6. The tropical modification $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ adds the ray $\sigma_{\emptyset<Z} = \sum_{i \in Z} e_i$ and subdivides each cone $\sigma_{Z<\mathcal{F}}$ into the cones $\{\sigma_{I<Z\cup\mathcal{F}} : I \subsetneq Z\}$. In other words, the tropical modification is a stellar subdivision, and $X(\Sigma_{\mathrm{M},\mathcal{P}_+})$ is the blow up of $X(\Sigma_{\mathrm{M},\mathcal{P}_-})$ along the torus-invariant subvariety $V(\sigma_{Z<\emptyset})$.

Example 5.4. Consider now a general matroidal flip $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ with center Z.

The tropical modification $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ adds the ray $\sigma_{\emptyset<Z} = \sum_{i \in Z} e_i$ and replaces each cone $\sigma_{Z<\mathcal{F}}$ with the cones $\{\sigma_{I<Z\cup\mathcal{F}} : I \subset Z \text{ and } I \text{ does not span } Z\}$. In the case

M^Z is not boolean, the support of the cone $\sigma_{Z<\mathcal{F}}$ strictly contains the union of the supports of the cones $\{\sigma_{I<Z\cup\mathcal{F}} : I \subset Z \text{ and } I \text{ does not span } Z\}$. In particular, $X(\Sigma_{\mathrm{M},\mathcal{P}_+})$ is not a blow up of $X(\Sigma_{\mathrm{M},\mathcal{P}_-})$.

However, the map $X(\Sigma_{\mathrm{M},\mathcal{P}_+}) \to X(\Sigma_{\mathrm{M},\mathcal{P}_-})$ does factor through the blow up of $X(\Sigma_{\mathrm{M},\mathcal{P}_-})$ along $V(\sigma_{Z<\emptyset})$. Specifically, $X(\Sigma_{\mathrm{M},\mathcal{P}_+})$ is an open subset of the blow up, obtained by removing the closed subvarieties

$$\{V(\sigma_{I<Z\cup\mathcal{F}}) : I \subset Z \text{ and } I \text{ spans } Z\}.$$

We now construct a suitable subdivision of the fans $\Sigma_{\mathrm{M},\mathcal{P}_+}$ and $\Sigma_{\mathrm{M},\mathcal{P}_-}$ in order to induce suitable Mayer–Vietoris sequences to which we will apply Lemma 5.2.

Definition 5.5. For a matroidal flip $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ with center Z, define the subfans

$$\Pi_+ = \{\sigma_{I<\mathcal{F}} : I \subsetneq Z \text{ and } Z < \mathcal{F}\} \subseteq \Sigma_{\mathrm{M},\mathcal{P}_+},$$

$$H_+ = \{\sigma_{I<\mathcal{F}} : Z \notin \mathcal{F}\} \subseteq \Sigma_{M,\mathcal{P}_+},$$

$$\Pi_- = \{\sigma_{I<\mathcal{F}} : I \subseteq Z \text{ and } Z < \mathcal{F}\} \subseteq \Sigma_{\mathrm{M},\mathcal{P}_-},$$

$$H_- = \{\sigma_{I<\mathcal{F}} : I \neq Z\} \subseteq \Sigma_{M,\mathcal{P}_-}.$$

Define the bigraded ring

$$\mathrm{E}^Z_{t,s} = \bigoplus_{\substack{t_1+t_2=t\\ s_1+s_2=s\\ t_1+s_1>0}} \mathrm{Tor}^{S^\circ_{\mathrm{M}^Z}}_{t_1}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \mathrm{Tor}^{S^\circ_{\mathrm{M}_Z}}_{t_2}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_2}.$$

The open toric subvarieties $X(\Pi_+)$ and $X(H_+)$ in $X(\Sigma_{\mathrm{M},\mathcal{P}_+})$ play the roles of the tubular neighborhood of the exceptional divisor and the complement of the exceptional divisor. The intersection of the two open sets is

$$X(\Pi_+) \cap X(H_+) = X(\Pi_+ \cap H_+),$$

where $\Pi_+ \cap H_+$ is the intersection of subfans of $\Sigma_{\mathrm{M},\mathcal{P}_+}$ and therefore a subfan itself. Similarly, $X(\Pi_-)$ and $X(H_-)$ in $X(\Sigma_{\mathrm{M},\mathcal{P}_-})$ play the roles of the small neighborhood of the center and the complement of the center. Their intersection is $X(\Pi_- \cap H_-)$.

From the description of H_+ and H_-, one verifies that $H_+ = H_-$ and $\Pi_+ \cap H_+ = \Pi_- \cap H_-$ as fans in $\mathbb{R}^n/\mathbb{R} \cdots (1,\dots,1)$. See Figure 2 for an example.

In terms of Stanley–Reisner rings, these decompositions of $\Sigma_{\mathrm{M},\mathcal{P}_+}$ and $\Sigma_{\mathrm{M},\mathcal{P}_-}$ yield the short exact sequences of S°_{M}-algebras

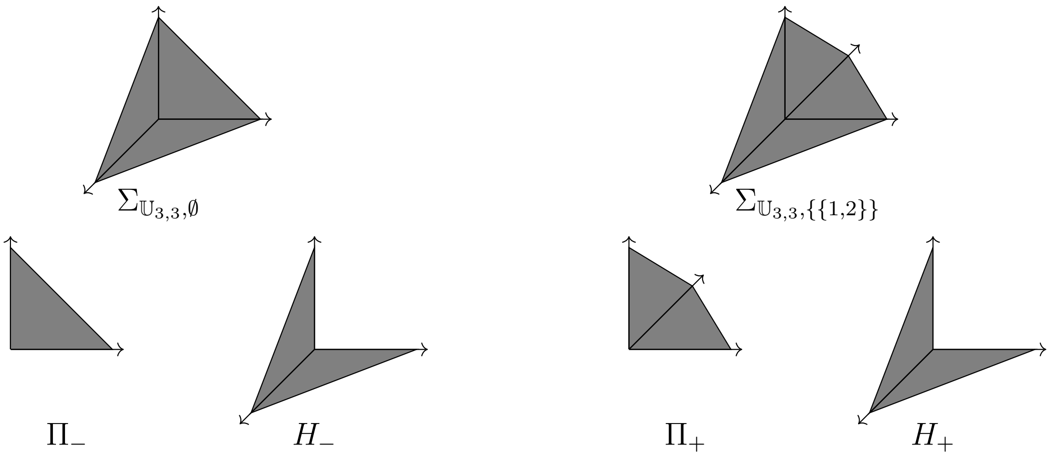

FIGURE 2. An example of a matroidal flip and subdivision of the Bergman fans.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}] & \longrightarrow & \frac{\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]}{I_{\Pi_+}\cap\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]} \oplus \frac{\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]}{I_{H_+}\cap\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]} & \longrightarrow & \frac{\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]}{I_{\Pi_+}+I_{H_+}\cap\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}] & \longrightarrow & \mathbb{C}[\Pi_+]\oplus\mathbb{C}[H_+] & \longrightarrow & \mathbb{C}[\Pi_+\cap H_+] & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}] & \longrightarrow & \mathbb{C}[\Pi_-]\oplus\mathbb{C}[H_-] & \longrightarrow & \mathbb{C}[\Pi_-\cap H_-] & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}] & \longrightarrow & \frac{\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]}{I_{\Pi_-}\cap\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]} \oplus \frac{\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]}{I_{H_-}\cap\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]} & \longrightarrow & \frac{\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]}{I_{\Pi_-}+I_{H_-}\cap\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]} & \longrightarrow & 0
\end{array}$$

Recall our definition of $\delta_i x_i \in \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]$ given by

$$\delta_i x_i = \begin{cases} x_i & \text{if } x_i \in \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}] \\ 0 & \text{otherwise.} \end{cases}$$

The map of S°_{M}-algebras

$$\iota\colon \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}] \to \mathcal{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}],$$

defined by $\iota(x_i) = \delta_i x_i + x_Z$ for $i \in Z$ and $\iota(x) = x$ for any other indeterminant, defines a map between the two exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}] & \longrightarrow & \mathbb{C}[\Pi_+]\oplus\mathbb{C}[H_+] & \longrightarrow & \mathbb{C}[\Pi_+\cap H_+] & \longrightarrow & 0 \\
 & & \iota\uparrow & & \iota\uparrow & & \iota\uparrow & & \\
0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}] & \longrightarrow & \mathbb{C}[\Pi_-]\oplus\mathbb{C}[H_-] & \longrightarrow & \mathbb{C}[\Pi_-\cap H_-] & \longrightarrow & 0
\end{array}$$

making the diagram commute. Because $x_Z = 0$ in $\mathbb{C}[H_+] = \mathbb{C}[H_-]$ and $\mathbb{C}[\Pi_+ \cap H_+] = \mathbb{C}[\Pi_- \cap H_-]$, the map ι is the identity map between these pairs.

Applying the functors $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}(-,\mathbb{C})_\bullet$ produces the analogue of the Mayer–Vietoris long exact sequence in each row.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_\bullet & \longrightarrow & \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_\bullet \oplus \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[H_+]\right)_\bullet & \longrightarrow & \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+\cap H_+]\right)_\bullet & \longrightarrow & \cdots \\
 & & \iota\uparrow & & \iota\uparrow \qquad\qquad \| & & \| & & \\
\cdots & \longrightarrow & \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_\bullet & \longrightarrow & \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_-]\right)_\bullet \oplus \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[H_-]\right)_\bullet & \longrightarrow & \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_-\cap H_-]\right)_\bullet & \longrightarrow & \cdots
\end{array}$$

In order to compare $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_\bullet$ with $\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_\bullet$ via Lemma 5.2, it then suffices to show

$$\iota\colon \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_-]\right)_\bullet \to \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_\bullet$$

is an injection.

Lemma 5.6. *There are isomorphisms*

$$\mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_s \cong \bigoplus_{\substack{t_1+t_2=t\\ s_1+s_2=s}} \mathrm{Tor}_{t_1}^{S^\circ_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \mathrm{Tor}_{t_2}^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_2}$$

and

$$\mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_-]\right)_s \cong 1 \otimes \mathrm{Tor}_t^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{M_Z}]\right)_s$$

such that ι is the natural inclusion.

In particular,

$$\iota\colon \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_-]\right)_\bullet \to \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_\bullet$$

is injective, and

$$\mathrm{coker}(\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_-]\right)_\bullet \to \mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_\bullet) \cong \mathrm{E}^Z_{\bullet,\bullet}.$$

Proof. From the definition of Π_+, we have the ring isomorphism

$$\mathbb{C}[\Pi_+] \cong \mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}] \otimes_{\mathbb{C}} \mathbb{C}[x_Z] \otimes_{\mathbb{C}} \mathbb{C}[\Sigma_{\mathrm{M}_Z}].$$

The maps

$$S^\circ_{\mathrm{M}^Z} \to \mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}],\ \mathbb{C}[x_Z] = \mathbb{C}[x_Z],\ \text{and } S^\circ_{\mathrm{M}_Z} \to \mathbb{C}[\Sigma_{\mathrm{M}_Z}]$$

give $\mathbb{C}[\Pi_+]$ the structure of an $S := S^\circ_{\mathrm{M}^Z} \otimes_{\mathbb{C}} \mathbb{C}[x_Z] \otimes_{\mathbb{C}} S^\circ_{\mathrm{M}_Z}$-algebra. Applying Lemma 2.10 twice,

$$\mathrm{Tor}_t^{S}\left(\mathbb{C}[\Pi_+]\right)_s \cong \bigoplus_{\substack{t_1+t_2+t_3=t\\ s_1+s_2+s_3=s}} \mathrm{Tor}_{t_1}^{S^\circ_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \mathrm{Tor}_{t_2}^{\mathbb{C}[x_Z]}\left(\mathbb{C}[x_Z]\right)_{s_2} \otimes \mathrm{Tor}_{t_3}^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_3}.$$

As

$$\mathrm{Tor}_\bullet^{\mathbb{C}[x_Z]}\left(\mathbb{C}[x_Z]\right)_\bullet = \mathrm{Tor}_0^{\mathbb{C}[x_Z]}\left(\mathbb{C}[x_Z]\right)_0 \cong \mathbb{C},$$

$$\mathrm{Tor}_t^{S}\left(\mathbb{C}[\Pi_+]\right)_s \cong \bigoplus_{\substack{t_1+t_2=t\\ s_1+s_2=s}} \mathrm{Tor}_{t_1}^{S^\circ_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \mathrm{Tor}_{t_2}^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_2}.$$

To show the same decomposition holds for $\mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_s$, we will construct a commutative square

$$\begin{array}{ccc} S & \xrightarrow{\phi_+} & S^\circ_{\mathrm{M}} \\ \downarrow & & \downarrow \\ \mathbb{C}[\Pi_+] & \xrightarrow{\psi_+} & \mathbb{C}[\Pi_+] \end{array}$$

such that the vertical maps define the S- and S°_{M}-algebra structures of $\mathbb{C}[\Pi_+]$ and such that the horizontal maps are degree-preserving isomorphisms. This yields a degree-preserving isomorphism

$$\mathrm{Tor}_\bullet^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_\bullet \cong \mathrm{Tor}_\bullet^{S}\left(\mathbb{C}[\Pi_+]\right)_\bullet,$$

and the decomposition will follow. Let $a = \max\{i \in Z\}$ and $b = \max\{i \in [n] \setminus Z\}$. The maps ϕ_+ and ψ_+ in the commutative square are given by

$$\phi_+(x) = \begin{cases} x_a - x_b & x = x_Z \\ x_i - x_j & x = x_i - x_j \text{ with } i, j \in Z \\ x_i - x_j & x = x_i - x_j \text{ with } i, j \notin Z. \end{cases}$$

and

$$\psi(x) = \begin{cases} \delta_a x_a + \sum_{\{F : a \in F\}} x_F - \delta_b x_b - \sum_{\{G : b \in G\}} x_G & x = x_Z \\ x & \text{other indeterminants.} \end{cases}$$

Therefore

$$\mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Pi_+]\right)_s \cong \bigoplus_{\substack{t_1+t_2=t \\ s_1+s_2=s}} \mathrm{Tor}_{t_1}^{S^\circ_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \mathrm{Tor}_{t_2}^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_2}.$$

We now apply a similar technique to $\mathbb{C}[\Pi_-]$. From the definition of Π_-, we have the ring isomorphism

$$\mathbb{C}[\Pi_-] \cong S_{\mathrm{M}^Z} \otimes_{\mathbb{C}} \mathbb{C}[\Sigma_{\mathrm{M}_Z}].$$

The maps

$$S_{\mathrm{M}^Z} = S_{\mathrm{M}^Z} \text{ and } S^\circ_{\mathrm{M}_Z} \to \mathbb{C}[\Sigma_{\mathrm{M}_Z}]$$

make $\mathbb{C}[\Pi_-]$ a $R := S_{\mathrm{M}^Z} \otimes_{\mathbb{C}} S^\circ_{\mathrm{M}_Z}$-algebra. Applying Lemma 2.10 and noting $\mathrm{Tor}_\bullet^{S_{\mathrm{M}^Z}}\left(S_{\mathrm{M}^Z}\right)_\bullet = \mathrm{Tor}_0^{S_{\mathrm{M}^Z}}\left(S_{\mathrm{M}^Z}\right)_0 \cong \mathbb{C}$,

$$\mathrm{Tor}_t^{R}\left(\mathbb{C}[\Pi_-]\right)_s \cong 1 \otimes \mathrm{Tor}_t^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_s.$$

Similar to the argument for $\mathbb{C}[\Pi_+]$, we have a commutative square

$$\begin{array}{ccc} R & \xrightarrow{\phi_-} & S^\circ_{\mathrm{M}} \\ \downarrow & & \downarrow \\ \mathbb{C}[\Pi_-] & \xrightarrow{\psi_-} & \mathbb{C}[\Pi_-] \end{array}$$

with vertical maps defining the algebra structures and the horizontal maps degree-preserving isomorphisms. Again taking $b = \max\{i \in [n] \setminus Z\}$,

$$\phi_-(x) = \begin{cases} x_i - x_b & x = x_i \text{ where } i \in Z \\ x_i - x_j & x = x_i - x_j \text{ with } i, j \notin Z. \end{cases}$$

and

$$\psi_-(x) = \begin{cases} \delta_i x_i + \sum_{\{F: i \in F\}} x_F - \delta_b x_b - \sum_{\{G: b \in G\}} x_G & x = x_i \text{ where } i \in Z \\ x & \text{other indeterminants.} \end{cases}$$

Therefore

$$\operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Pi_-]\right)_s \cong \operatorname{Tor}_t^{R} \left(\mathbb{C}[\Pi_-]\right)_s \cong 1 \otimes \operatorname{Tor}_t^{S^\circ_{\mathrm{M}_Z}} \left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_s.$$

Each of these isomorphisms can be realized as an isomorphism on the cohomology of Koszul complexes. In particular, the natural inclusion

$$\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}} \hookrightarrow \mathcal{K}^\bullet(\mathbb{C}[\Pi_-])_{S^\circ_{\mathrm{M}}}$$

is a quasi-isomorphism, as is the natural inclusion of the tensor complex

$$\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}])_{S^\circ_{\mathrm{M}^Z}} \otimes_{\mathbb{C}} \mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}} \hookrightarrow \mathcal{K}^\bullet(\mathbb{C}[\Pi_+])_{S^\circ_{\mathrm{M}}}.$$

The map $\iota\colon \mathbb{C}[\Pi_-] \to \mathbb{C}[\Pi_+]$ induces the map $\iota\colon \mathcal{K}^\bullet(\mathbb{C}[\Pi_-])_{S^\circ_{\mathrm{M}}} \to \mathcal{K}^\bullet(\mathbb{C}[\Pi_+])_{S^\circ_{\mathrm{M}}}$, which when restricted to $\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}}$ is the inclusion

$$\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}} \hookrightarrow \mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}])_{S^\circ_{\mathrm{M}^Z}} \otimes_{\mathbb{C}} \mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}}.$$

Therefore, the identifications

$$\operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Pi_+]\right)_s \cong \bigoplus_{\substack{t_1+t_2=t \\ s_1+s_2=s}} \operatorname{Tor}_{t_1}^{S^\circ_{\mathrm{M}^Z}} \left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \operatorname{Tor}_{t_2}^{S^\circ_{\mathrm{M}_Z}} \left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_2}$$

and

$$\operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Pi_-]\right)_s \cong 1 \otimes \operatorname{Tor}_t^{S^\circ_{\mathrm{M}_Z}} \left(\mathbb{C}[\Sigma_{M_Z}]\right)_s$$

realize

$$\iota\colon \operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Pi_-]\right)_s \to \operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Pi_+]\right)_s$$

as the natural inclusion. □

By applying Lemma 5.2 to the long exact sequence of Tor groups induced by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}] & \longrightarrow & \mathbb{C}[\Pi_+] \oplus \mathbb{C}[H_+] & \longrightarrow & \mathbb{C}[\Pi_+ \cap H_+] & \longrightarrow & 0 \\ & & \iota\uparrow & & \iota\uparrow & & \iota\uparrow & & \\ 0 & \longrightarrow & \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}] & \longrightarrow & \mathbb{C}[\Pi_-] \oplus \mathbb{C}[H_-] & \longrightarrow & \mathbb{C}[\Pi_- \cap H_-] & \longrightarrow & 0, \end{array}$$

we get the long exact sequence of a matroidal flip.

Theorem 5.7. *For a matroidal flip $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ with center Z, there is a long exact sequence*

$$\cdots \to E^Z_{\bullet+1,\bullet-1} \to \operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_{\bullet} \to \operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_{\bullet} \to E^Z_{\bullet,\bullet} \to \cdots .$$

Remark 5.8. The long exact sequence of a matroidal flip is not induced by a short exact sequence of S°_{M}-algebras. In particular, the map $\iota\colon \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}] \to \mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]$ constructed in the proof above is not injective, and

$$\operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_{\bullet} \ncong \operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\operatorname{im}\iota\right)_{\bullet}.$$

In order to use this exact sequence to deduce information about $\operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_{\bullet}$ from $\operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_{\bullet}$ we need to understand the kernel of the connecting homomorphism $E^Z_{\bullet+1,\bullet-1} \to \operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_{\bullet}$.

Lemma 5.9. *Let* M *be a loopless matroid and $\mathcal{P}_+ \setminus \mathcal{P}_- = Z$ two order filters. Denote by Q^Z the subgroup of*

$$E^Z_{t,s} = \bigoplus_{\substack{t_1+t_2=t\\ s_1+s_2=s\\ t_1+s_1>0}} \operatorname{Tor}^{S^\circ_{\mathrm{M}^Z}}_{t_1}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{s_1} \otimes \operatorname{Tor}^{S^\circ_{\mathrm{M}_Z}}_{t_2}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{s_2}$$

where $t_1 = 0$ or where $t_1 = 1$ and $t_2 = 0$. Then the connecting homomorphism

$$E^Z_{\bullet+1,\bullet-1} \to \operatorname{Tor}^{S^\circ_{\mathrm{M}}}_{\bullet}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_{i-1}}]\right)_{\bullet}$$

of the long exact sequence of the matroidal flip vanishes on Q^Z.

Proof. We view the Tor groups as Koszul cohomology groups. Take $\psi \in Q$; by linearity of the connecting homomorphism, we may assume that ψ takes the form $\eta \otimes \xi$ where $\eta \in H^0(\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}])_{S^\circ_{\mathrm{M}^Z}})$ has Stanley–Reisner degree at least 1 and $\xi \in H^\bullet(\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}})$, or where $\eta \in H^1(\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}])_{S^\circ_{\mathrm{M}^Z}})$ and $\xi \in H^0(\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}})$.

In the first case, $H^0(\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}])_{S^\circ_{\mathrm{M}^Z}})$ is generated by monomials in the Stanley–Reisner ring, with two monomials of the same degree cohomologous (cf. [1, pg. 411]), so we may take $\eta = x_i^j$ for some $i \in Z$ and $j > 0$. Let Π_+ and H_+ be as in Definition 5.5. Viewing $\eta \otimes \xi \in \mathcal{K}^\bullet(\mathbb{C}[\Pi_+])_{S^\circ_{\mathrm{M}}}$, this is homologous to $x_Z^j \otimes \xi$, up to an element of $1 \otimes H^\bullet(\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M}_Z}])_{S^\circ_{\mathrm{M}_Z}})$. From the construction of the long exact sequence of a matroidal flip in Theorem 5.7 and Lemma 5.2, the connecting homomorphism factors through the map

$$\mathcal{K}^\bullet(\mathbb{C}[\Pi_+])_{S^\circ_{\mathrm{M}}} \oplus \mathcal{K}^\bullet(\mathbb{C}[H_+])_{S^\circ_{\mathrm{M}}} \to \mathcal{K}^\bullet(\mathbb{C}[\Pi_+ \cap H_+])_{S^\circ_{\mathrm{M}}}$$

which sends x_Z to 0. Therefore $\eta \otimes \xi$ is in the kernel of the connecting homomorphism.

In the second case, we may take η to be a basis element as in Lemma 4.7, so $\eta = \prod_{k \in B\setminus i_B} x_k \otimes (x_{i_B} - x_{j_B})$ for a non-maximal basis B of M^Z. Viewing $\eta \otimes \xi \in$

$\mathcal{K}^\bullet(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}])_{S^\circ_{\mathrm{M}}}$ and applying the differential

$$\begin{aligned} d(\eta\otimes\xi) = d\eta\otimes\xi &= \prod_{k\in B\setminus i_B} x_k \left(x_{i_B} + \sum_{\substack{F\in\mathcal{P}_+\\ i_B\in F}} x_F - x_{j_B} - \sum_{\substack{G\in\mathcal{P}_+\\ j_B\in G}} x_G \right)\otimes\xi \\ &= \prod_{k\in B\setminus i_B} x_k \left(x_{i_B} + \sum_{\substack{F\in\mathcal{P}_+\\ B\subseteq F}} x_F - x_{j_B} - \sum_{\substack{G\in\mathcal{P}_+\\ B\setminus i_B\cup j_B\subseteq G}} x_G \right)\otimes\xi \\ &= \prod_{k\in B\setminus i_B} x_k \left(x_{i_B} + \sum_{\substack{F\in\mathcal{P}_+\\ Z\subseteq F}} x_F - x_{j_B} - \sum_{\substack{G\in\mathcal{P}_+\\ Z\subseteq G}} x_G \right)\otimes\xi \end{aligned}$$

(as B and $B\setminus i_B\cup j_B$ span Z)

$$= \prod_{k\in B} x_k - \prod_{k\in B\setminus i_B\cup j_B} x_k = 0.$$

This implies $\eta\otimes\xi$ is in the image of the map

$$\mathrm{Tor}^{S^\circ_{\mathrm{M}}}_\bullet\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_\bullet \to E^Z_{\bullet,\bullet}$$

and therefore vanishes under the connecting homomorphism by exactness. □

6. Applications of the Long Exact Sequence of a Matroidal Flip

We now deduce some properties of $\mathrm{Tor}^{S^\circ_{\mathrm{M}}}_\bullet(\mathbb{C}[\Sigma_{\mathrm{M}}])_\bullet$ from those of $\mathrm{Tor}^{S^\circ_{\mathrm{M}}}_\bullet(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet$ by using the long exact sequence associated to matroidal flips. In particular, we show the sharp vanishing result for $\mathrm{Tor}^{S^\circ_{\mathrm{M}}}_\bullet(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_\bullet$ carries over to $\mathrm{Tor}^{S^\circ_{\mathrm{M}}}_\bullet(\mathbb{C}[\Sigma_{\mathrm{M}}])_\bullet$, which is Theorem 6.1, and we get a complete description of

$$\mathrm{Hilb}\left(\mathrm{Tor}^{S^\circ_{\mathbb{U}_{r,k+r}}}_\bullet\left(\mathbb{C}[\Sigma_{\mathbb{U}_{r,k+r}}]\right)_\bullet\right),$$

which is Theorem 6.5.

Theorem 6.1. *Let* M *be a loopless matroid of rank $r>0$ on the ground set $[n]$. For $t>n-r$ or $s>r-1$, $\mathrm{Tor}^{S^\circ_{\mathrm{M}}}_t(\mathbb{C}[\Sigma_{\mathrm{M}}])_s = 0$.*

Moreover in top degree

$$\dim \mathrm{Tor}^{S^\circ_{\mathrm{M}}}_{n-r}(\mathbb{C}[\Sigma_{\mathrm{M}}])_{r-1} = |\mathrm{nbc}(\mathrm{M})| \neq 0.$$

Proof. Choose a sequence of matroidal flips

$$\Sigma_{\mathrm{M},\emptyset} = \Sigma_{\mathrm{M},\mathcal{P}_0} \rightsquigarrow \cdots \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_i} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_{i+1}} \rightsquigarrow \cdots \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_q} = \Sigma_{\mathrm{M}}.$$

We make a double induction, first inducting on the rank of M, the base case where $r=1$ being Proposition 2.12, and then inducting on the index i of the sequence of order filters, the base case $i=0$ being covered in Corollary 4.6.

Suppose $Z = \mathcal{P}_i \setminus \mathcal{P}_{i-1}$. The induction on r shows that $E^Z_{t,s} = 0$ for $t > n - r$ or $s \geq r - 1$, as $\operatorname{Tor}_t^{S^\circ_{\mathrm{M}^Z}} \left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_s = 0$ for $t > |Z| - \operatorname{rank}_{\mathrm{M}}(Z)$ or $s > \operatorname{rank}_{\mathrm{M}}(Z) - 1$, and $\operatorname{Tor}_t^{S^\circ_{\mathrm{M}_Z}} \left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_s = 0$ for $t > n - |Z| - r + \operatorname{rank}_{\mathrm{M}}(Z)$ or $s > r - \operatorname{rank}_{\mathrm{M}}(Z) - 1$.

The long exact sequence of a matroidal flip gives the exactness of

$$E^Z_{t+1,s-1} \to \operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_{i-1}}]\right)_s \to \operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_i}]\right)_s \to E^Z_{t,s}.$$

In particular, the vanishing of $E^Z_{\bullet,\bullet}$ implies

$$\operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_i}]\right)_s \cong \operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_{i-1}}]\right)_s$$

for $t > n - r$ or $s > r - 1$, as well as the case $t = n - r$ and $s = r - 1$.

From the induction on i indexing the order filters,

$$\operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_i}]\right)_s = \operatorname{Tor}_t^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_{i-1}}]\right)_s = 0$$

for $t > n - r$ or $s > r - 1$.

If M is moreover loopless, this same induction shows

$$\dim \operatorname{Tor}_{n-r}^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_i}]\right)_{r-1} = \dim \operatorname{Tor}_{n-r}^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_{i-1}}]\right)_{r-1} = |\operatorname{nbc}(\mathrm{M})|.$$

For loopless matroids, the lex-minimal basis is always a no-broken-circuit basis, so $|\operatorname{nbc}(\mathrm{M})| > 0$. ☐

Remark 6.2. In general, we have been unable to use the long exact sequences of matroidal flips to completely deduce the Hilbert series of $\operatorname{Tor}_\bullet^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma]\right)_\bullet$. This is because determining the kernel of the connecting homomorphism in the long exact sequence of a matroidal flip is a difficult task, even in the case of rank 3 matroids. In particular, the subgroup Q^Z in Lemma 5.9 is not always the entire kernel, nor does the connecting homomorphism always vanish.

For example, take the matroid M on [6] with bases

$$\mathcal{B}(\mathrm{M}) = \{125, 126, 135, 136, 145, 146, 235, 236, 245, 246, 256, 345, 346, 356, 456\}.$$

First assume that for every matroidal flip with center Z, the subgroup Q^Z were the entire kernel of the connecting homomorphism. Then $\operatorname{Hilb}\left(\operatorname{Tor}_\bullet^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M}}]\right)_\bullet\right)$ would be

$$1 + 9y + y^2 + 28xy + 2xy^2 + 19x^2y + 12x^2y^2 + 6x^3y + 6x^3y^2.$$

On the other hand, if we assume that for every matroidal flip the connecting homomorphism were to vanish, then $\operatorname{Hilb}\left(\operatorname{Tor}_\bullet^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M}}]\right)_\bullet\right)$ would be

$$1 + 9y + y^2 + 28xy + 14xy^2 + 31x^2y + 17x^2y^2 + 11x^3y + 6x^3y^2.$$

However, neither of these assumptions produce the correct result. A direct computation (using Macaulay2 and the Matroids package [6, 15]) yields

$$\operatorname{Hilb}\left(\operatorname{Tor}_\bullet^{S^\circ_{\mathrm{M}}} \left(\mathbb{C}[\Sigma_{\mathrm{M}}]\right)_\bullet\right) = 1 + 9y + y^2 + 28xy + 7xy^2 + 24x^2y + 13x^2y^2 + 7x^3y + 6x^3y^2.$$

There are two cases, however, where the kernel of the connecting homomorphism is fully known. The first is for the connecting homomorphism from Tor_1 to Tor_0, and the second is when the center Z of the matroidal flip makes M^Z a boolean matroid, as this corresponds to the case when the matroidal flip is precisely a blow up.

Corollary 6.3 (cf. [1, Theorem 6.18])**.** *Let* M *be a loopless matroid, and* $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ *a matroidal flip with center* Z*. Then there is a short exact sequence*

$$0 \to A^\bullet(\mathrm{M},\mathcal{P}_-) \to A^\bullet(\mathrm{M},\mathcal{P}_+) \to A^{>0}(\mathrm{M}^Z,\emptyset) \otimes A^\bullet(\mathrm{M}^Z) \to 0.$$

Proof. By Theorem 5.7, there is an exact sequence

$$\mathrm{E}^Z_{1,\bullet-1} \to \mathrm{Tor}_0^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_\bullet \to \mathrm{Tor}_0^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_\bullet \to \mathrm{E}^Z_{0,\bullet} \to 0.$$

Now $\mathrm{E}^Z_{1,\bullet}$ is defined to be

$$\mathrm{Tor}_0^{S^\circ_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{>0} \otimes \mathrm{Tor}_1^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_\bullet \oplus \mathrm{Tor}_1^{S^\circ_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_\bullet \otimes \mathrm{Tor}_0^{S^\circ_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_\bullet.$$

Therefore $\mathrm{E}^Z_{1,\bullet} = Q^Z$, so the map $\mathrm{E}^Z_{1,\bullet-1} \to \mathrm{Tor}_0^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_\bullet$ vanishes by Lemma 5.9. We are left with the short exact sequence

$$0 \to \mathrm{Tor}_0^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_\bullet \to \mathrm{Tor}_0^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_\bullet \to \mathrm{E}^Z_{0,\bullet} \to 0,$$

which is precisely

$$0 \to A^\bullet(\mathrm{M},\mathcal{P}_-) \to A^\bullet(\mathrm{M},\mathcal{P}_+) \to A^{>0}(\mathrm{M}^Z,\emptyset) \otimes A^\bullet(\mathrm{M}^Z) \to 0.$$

□

Corollary 6.4. *If* M *is a uniform matroid, then for each matroidal flip* $\Sigma_{\mathrm{M},\mathcal{P}_-} \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_+}$ *with center* Z*, and for each* $t \geq 0$ *there is a short exact sequence*

$$0 \to \mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_\bullet \to \mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_\bullet \to \mathrm{E}^Z_{t,\bullet} \to 0.$$

Proof. Note that M^Z is a boolean matroid. By Corollary 4.6, $\mathrm{Tor}_t^{S^\circ_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_\bullet$ is non-zero only in $t = 0$. This implies $\mathrm{E}^Z_{\bullet,\bullet} = Q^Z$. By Lemma 5.9, the connecting homomorphism of the long exact sequence of the matroidal flip vanishes. Therefore the long exact sequence breaks into short exact sequences

$$0 \to \mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_-}]\right)_\bullet \to \mathrm{Tor}_t^{S^\circ_{\mathrm{M}}}\left(\mathbb{C}[\Sigma_{\mathrm{M},\mathcal{P}_+}]\right)_\bullet \to \mathrm{E}^Z_{t,\bullet} \to 0$$

for each $t \geq 0$. □

We now recall and prove Theorem 6.5.

Theorem 6.5. *Let* r *and* k *be positive integers. Then*

$$\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S^\circ_{\mathbb{U}_{r,k+r}}}\left(\mathbb{C}[\Sigma_{\mathbb{U}_{r,k+r}}]\right)_\bullet\right) = \mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S^\circ_{\mathbb{U}_{r,k+r}}}\left(\mathbb{C}[\Sigma_{\mathbb{U}_{r,k+r}},\emptyset]\right)_\bullet\right) + \sum_{i=1}^{r-1}\binom{r+k}{i}\left(\frac{y-y^i}{1-y}\right)\mathrm{Hilb}\left(\mathrm{Tor}_\bullet^{S^\circ_{\mathbb{U}_{r-i,r+k-i}}}\left(\mathbb{C}[\Sigma_{\mathbb{U}_{r-i,r+k-i}}]\right)_\bullet\right).$$

Proof. Let $\mathrm{M} = \mathbb{U}_{r,k+r}$. Take a sequence of matroidal flips

$$\Sigma_{\mathrm{M},\emptyset} = \Sigma_{\mathrm{M},\mathcal{P}_0} \rightsquigarrow \cdots \rightsquigarrow \Sigma_{\mathrm{M},\mathcal{P}_q} = \Sigma_{\mathrm{M}}.$$

Iteratively applying the short exact sequences in Corollary 6.4 produces

$$\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M}}])_{\bullet}\right) = \mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S_{\mathrm{M}}^{\circ}}(\mathbb{C}[\Sigma_{\mathrm{M},\emptyset}])_{\bullet}\right) + \sum_{Z\in\widehat{\mathcal{L}}(\mathrm{M})} \mathrm{Hilb}(\mathrm{E}^{Z}_{\bullet,\bullet}).$$

The definition of $\mathrm{E}^{Z}_{\bullet,\bullet}$ implies

$$\mathrm{Hilb}(\mathrm{E}^{Z}_{\bullet,\bullet}) = \left[\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S^{\circ}_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{\bullet}\right) - 1\right] \cdot \mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S^{\circ}_{\mathrm{M}_Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}_Z}]\right)_{\bullet}\right)$$

because $\dim \mathrm{Tor}_0^{S^{\circ}_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_0 = 1$.

We see $\mathrm{M}^Z \cong \mathbb{U}_{i,i}$ and $\mathrm{M}^Z \cong \mathbb{U}_{r-i,r+k-i}$ for some integer i. In this case,

$$\mathrm{Hilb}\left(\mathrm{Tor}_{\bullet}^{S^{\circ}_{\mathrm{M}^Z}}\left(\mathbb{C}[\Sigma_{\mathrm{M}^Z,\emptyset}]\right)_{\bullet}\right) - 1 = \frac{y - y^i}{1-y}$$

by Theorem 4.4. There are precisely $\binom{r+k}{i}$ flats $Z \in \widehat{\mathcal{L}}(M)$ such that $\mathrm{M}^Z \cong \mathbb{U}_{i,i}$, and this concludes the proof. □

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